MATHEMATICS

PART - 1

Standard

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THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he.

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders, respect and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone, lies my happiness.

MATHEMATICS

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Dear children,

To understand the world and to recognise its peculiarities, all kinds of measurements and the relations between them were essential for humans.

We’ve seen how natural numbers and fractions, and illustrations with them evolved like this, in earlier classes. Here we will see certain measures which cannot be expressed using numbers learnt so far.

We also continue our study of geometry. This book discusses the new idea of similarity and how it ties up with the notion of parallel lines studied earlier.

Our new vision of learning mathematics emphasizes computational thinking, which gives greater emphasis to analysis of mathematical processes. What this means is that, instead of learning different mathematical techniques by rote and using them mechanically in specified contexts, it is more important to understand how and why they work and to use different methods to make them more efficient.

We also discuss the possibilities of the free software GeoGebra to make geometry more dynamic and thereby enhance comprehension.

With love and regards,

Dr. Jayaprakash R.K.
Director
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Certain icons are used in this textbook for convenience


Let's do problems


Project


ICT Possibilities

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

---

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

1 PAIRS OF EQUATIONS

Mental math and algebra

Let's start with a problem:

> There are 100 beads, black and white, in a box; 10 more black than white. How many black and how many white ?

This can be thought out in different ways.

If we keep the extra 10 black beads apart for the time being, there are 90 in the box, with black and white equal; which means 45 each. Now putting back the 10 black removed, there are 55 black and 45 white.

We can use a little bit of algebra (Remember the lesson, **Equations** in Class 8?)

> Taking the number of black beads as x, the number of white is $x - 10$. Since there are 100 altogether,

$$x + (x - 10) = 100$$

We can extract x from this:

$$2x - 10 = 100$$
$$2x = 110$$
$$x = 55$$

Thus we get the number of black beads as 55; and subtracting 10 from this, we can get the number of white as 45.

Let's look at another one:

> The price of a table and chair is 11000 rupees. The price of the table and four chairs is 14000 rupees. What's the price of each ?

First see if you can work this out in your head. The price of a table and four chairs is 3000 rupees more, because of three extra chairs, right? So, this extra 3000 is the price of 3 chairs; which means the price of a chair is 1000 rupees. And this gives the price of the table as 10000 rupees.

Instead of thinking like this, we can just start by taking the price of a chair as x rupees. A little thinking shows, the price of the table is then $11000 - x$ rupees. So the price of the table and four chairs is

$$(11000 - x) + 4x = 11000 + 3x$$

This is given to be 14000 rupees. That is,

$$11000 + 3x = 14000$$

And we can get x from this:

$$\begin{aligned} 11000 + 3x &= 14000 \\ 3x &= 3000 \\ x &= 1000 \end{aligned}$$

Thus we get the price of a chair as 1000 rupees and then the price of the table as $11000 - 1000 = 10000$ rupees.

There's another way. Let's take the price of the chair as x and the price of the table as y. Then what is given in the problem can be written as two equations:

$$\begin{aligned} x + y &= 11000 \\ 4x + y &= 14000 \end{aligned}$$

Now the relation between the numbers x and y given in the first equation, we can write like this:

$$y = 11000 - x$$

So, in the second equation, we can use $11000 - x$ in the place of y:

$$4x + (11000 - x) = 14000$$

That is,

$$11000 + 3x = 14000$$

This is the same old equation we got by taking only the price of a chair as x rupees, isn't it? So, we can compute the prices as before

Another problem:

> Of two numbers, the larger one is 5 times the smaller one; and subtracting the smaller from the larger gives 32. What are the numbers?

Can you do it in your head?

One number is five times the other; so what can we say instead of subtracting the smaller from the larger?

Subtracting the smaller number from five times itself, right?

Which means four times the number

This is 32, we are told. Thus 4 times the smaller number is 32, and so the number itself is 8.

Now we can compute the larger number as 5 times 8, which is 40.

How about using algebra?

Let's start by taking the smaller number as x, so that the larger is $5x$.

Subtracting the smaller from the larger, $5x - x = 4x$

Since the difference is 32, we have $4x = 32$

This gives $x = 8$

So, the smaller number is 8 and the larger, $5 \times 8 = 40$

What if we start by taking the smaller number as x and the larger y ?

We can write the information given in the problem as two equations:

$$y = 5x$$

$$y - x = 32$$

In the second equation, we can use $5x$ in the place of y, right?

$$5x - x = 32$$

That is,

$$4x = 32$$

From this we get $x = 8$ and then $y = 5 \times 8 = 40$.

One more problem:

> A fraction simplified after adding 1 to its numerator becomes $\frac{1}{2}$. If instead, 1 is added to the denominator and then simplified, it becomes $\frac{1}{3}$. What is the fraction ?

Can you do it in your head?

Even after taking only the numerator or denominator as x, we can't get far. Let's denote the numerator as x and denominator as y, so that the fraction is $\frac{x}{y}$.

Now let's write each bit of information given, as an equation:

$$\frac{x+1}{y} = \frac{1}{2}$$

$$\frac{x}{y+1} = \frac{1}{3}$$

What does the first equation tell us ?

The fraction $\frac{x+1}{y}$ is another form of $\frac{1}{2}$

In all the various forms of $\frac{1}{2}$ the denominator is twice the numerator, isn't it?

So,

$$y = 2(x + 1)$$

What about the second equation, $\frac{x}{y+1} = \frac{1}{3}$?

As before, this gives,

$$y + 1 = 3x$$

The first equation says, the numbers y and $2(x + 1)$ are equal. So, we can write $2(x + 1)$ instead of y in the second equation.

This gives

$$2(x + 1) + 1 = 3x$$

That is,

$$2x + 2 + 1 = 3x$$

$$2x + 3 = 3x$$

From this, we get $x = 3$. Then from the first equation, we get $y = 2 \times (3 + 1) = 2 \times 4 = 8$

Thus the fraction in question is $\frac{3}{8}$

Now try these problems in any way you like: as mental math, as an equation with a single letter or as a pair of equations with two letters:

(1) Priya bought a bag and a pair of slippers for 1100 rupees. The bag costs 300 rupees more than the slippers. What is the price of the slippers? And the price of the bag?

(2) The sum of two numbers is 26 and their difference is 4. What are the numbers ?

(3) The perimeter of a rectangle is 40 centimetres, and one side is 8 centimetres longer than the other. Calculate the lengths of the sides.

(4) A wire three and a half metres long is to be cut into two pieces, with one piece bent into a square and the other into an equilateral triangle. The lengths of the sides of both must be the same. How should the wire be cut ?

(5) In a class, there are 4 more girls than boys. On a day when only 8 boys were absent, the number of girls was twice the number of boys. How many boys and girls are there in the class ?

(6) A fraction simplified after adding 1 to its numerator becomes $\frac{1}{3}$. If instead, 1 is added to the denominator and then simplified, it becomes $\frac{1}{4}$. What is the fraction?

(7) A person invested 100000 rupees in two schemes, with interest rates 7% and 6%. After one year, they got 6750 rupees as interest from both these together. How much did they invest in each scheme?

(8) An object starts with a speed of u m/s and travels along a straight line. If the speed increases at the rate of a m/s every second, the speed at time t seconds is $u + at$. The speed at one second is 5 m/s and at five seconds, 13 m/s. What is the rate at which speed is increasing ? What was the starting speed?

Two equations

See this problem:

The price of 2 pens and 3 notebooks is 110 rupees. The price of 2 pens and 5 notebooks is 170 rupees. What is the price of a pen? And a notebook?

Think as we did in the table and chair problem. How did the cost increase from 110 rupees to 170 rupees ? Because of two more notebooks, right?

So, the 60 rupees increase is the price of 2 extra notebooks; and this means the price of a notebook is 30 rupees. Now to get the price of two pens from the first statement, we need only subtract the price of three note books from 110

Information and solutions

If we are only told that a box contains ten beads, black and white, we cannot definitely say how many of each. It can be just one black and nine white, two black and eight white and so on. If we are also told that there are two more black than white, we can fix the numbers as six black and four white.

In the language of algebra, there are many pairs of numbers x and y satisfying the single equation $x + y = 10$. But there is only one pair satisfying both the equations

$$x + y = 10$$
$$x - y = 2$$

and these are $x = 6$, $y = 4$

This gives the price of two pens as 110 – 90 = 20 rupees and the price of one pen as 10 rupees.

Now let's see how we do this by taking the price of a pen as x rupees, the price of a notebook as y rupees and converting the pieces of information given into algebraic equations:

Price of 2 pens and 3 notebooks is 110 rupees	$2x + 3y = 110$
Price of 2 pens and 5 notebooks is 170 rupees	$2x + 5y = 170$
Increase is the price of 2 notebooks	$(2x + 5y) - (2x + 3y) = 2y$
Increase is 60 rupees	$170 - 110 = 60$
Price of two notebooks is 60 rupees	$2y = 60$
Price of a notebook is 30 rupees	$y = 30$
Price of 2 pens is 90 rupees subtracted from 110 rupees	$2x = 110 - (3 \times 30) = 20$
Price of a pen is 10 rupees	$x = 10$

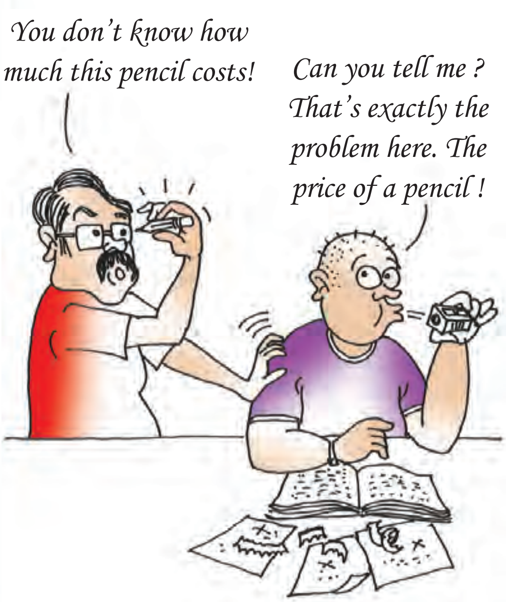
Let's do a slightly different problem:

> The price of 3 pencils and 4 pens is 66 rupees; and the price of 6 pencils and 3 pens is 72 rupees. What is the price of each ?

Let's first see if this can be done in head. Here the increase in price is not due to the increase in the number of just one thing, as in the first problem. So, things are not that easy.

If the number of pencils or pens was the same in both occasions, we could've done it as before. How about making it the same ?

Let's write the prices like this:



Pencil	Pen	Price
3	4	66
6	3	72

The number of pencils is 3 in the first line and 6 in the second. Can we make it 6 in the first line also? How about 6 pencils and 8 pens?

	Pencil	Pen	Price
	3	4	66
×2	6	3	72
	6	8	132

From the second line to the third, how much did the price increase ? Why?

The 60 rupees increase is the price of 5 pens only, right ?

So, the price of a pen is 12 rupees.

Now from the first line, we can compute the price of 3 pencils as 66 – 48 = 18 and then the price of a pencil as 6 rupees

Now let's put this all in algebra. Taking the price of a pencil as x rupees, that of a pen as y rupees, we can write the information given in the problem and the method of solution like this:

Price of 3 pencils and 4 pens is 66 rupees.

$$3x + 4y = 66$$

Price of 6 pencils and 3 pens is 72 rupees

$$6x + 3y = 72$$

Price of 6 pencils and 8 pens is 132 rupees

$$6x + 8y = 2(3x + 4y) = 132$$

Increase is the price of 5 pens

$$(6x + 8y) - (6x + 3y) = 5y$$

Increase is 60 rupees

$$132 - 72 = 60$$

Price of 5 pens is 60 rupees

$$5y = 60$$

Price of a pen is 12 rupees

$$y = 12$$

Price of 3 pencils is price of 4 pens subtracted from 66 rupees

$$3x = 66 - (4 \times 12) = 18$$

Price of a pencil is 6 rupees

$$x = 6$$

We can write all this in a compressed form. Let's write the pieces of information given in the problem as equations and number them:

$$3x + 4y = 66 \qquad (1)$$

$$6x + 3y = 72 \qquad (2)$$

Nothing new

Ramu bought a pencil and a pen for 7 rupees. Aju bought 4 pencils and 4 pens for 28 rupees. They tried to work out the price of each using these pieces of information.

Taking the price of a pencil as x rupees. they wrote the price of a pen as $7 - x$, using Ramu's purchase. Then made an equation of Aju's purchase:

$$4x + 4(7 - x) = 28$$

And what did they get on simplification?

$$28 = 28$$

What if they had taken the price of a pencil as x rupees and the price of a pen as y rupees?

$$x + y = 7$$

$$4x + 4y = 28$$

If we write the second equation as

$$4(x + y) = 28$$

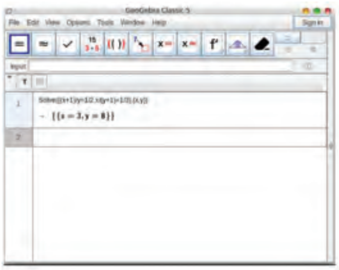
we again get

$$x + y = 7$$

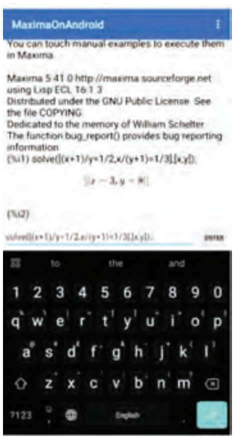
Thus in this problem, there's actually just one piece of information connecting the prices, though they're stated somewhat differently. And we can't compute the prices using this alone.

Computer algebra

Computers can be used not only to do complicated numerical computations, but also to solve algebraic problems. Such software are collectively called Computer Algebra System (CAS) or Symbolic Algebra System. Two prominent free software in this category are Sage Math and Maxima. GeoGebra also has some CAS capabilities:



GeoGebra and Maxima are available in Android phones also:



We can use the CAS in GeoGebra to solve a pair of equations. For example, to solve the equations, $5x + 2y = 20$, $2x + 3y = 19$ open the CAS window by

View → CAS and type

Solve ({$5x + 2y = 20$, $2x + 3y = 19$}, {x, y})

Equation (1) says the number $3x + 4y$ is 66. So, twice that is 132:

$$6x + 8y = 132 \qquad (3)$$

Now we can use Equations (2) and (3) to write

$$(6x + 8y) - (6x + 3y) = 132 - 72$$

Simplified, this gives

$$5y = 60$$

and from this, we get $y = 12$

Now putting $y = 12$ in Equation (1), we can compute x:

$$3x + (4 \times 12) = 66$$

$$3x = 66 - 48 = 18$$

$$x = 6$$

Now try these:

(1) The price of 4 pens and 3 pencils is 66 rupees. The price of 7 pens and 3 pencils is 111 rupees. What is the price of a pen? The price of a pencil ?

(2) The perimeter of a rectangle is 26 centimetres. Another rectangle with twice the length and thrice the breadth has perimeter 62 centimetres. What are the length and breadth of the first rectangle ?

Let's do another problem:

When a small vessel was filled and emptied five times and a big one two times into a bucket, it contains 20 litres. Instead when this was done twice with the small vessel and thrice with the big, it contained only 19 litres. How much can each vessel hold ?

Let's take the capacity of the small vessel as x litres and the capacity of the big one as y litres, to convert the given pieces of information into equations:

$$5x + 2y = 20 \qquad (1)$$
$$2x + 3y = 19 \qquad (2)$$

As in the last problem, to change $5x$ to $2x$ in Equation (1), we'll have to multiply by $\frac{2}{5}$; on the other hand, to change $2x$ to $5x$ in Equation (2), we'll have to multiply by $\frac{5}{2}$. We can find the solutions either way.

Is there a way to avoid fractions?

We can multiply the equations by any numbers. How about multiplying the first equation by one number and multiplying the second by a different number to make the multiplier of x the same in both?

$$(1) \times 2 : 10x + 4y = 40 \qquad (3)$$
$$(2) \times 5 : 10x + 15y = 95 \qquad (4)$$

Now we can subtract (3) from (4) to get

$$11y = 55$$

which gives

$$y = 5$$

And then we can use this in (1) to find x:

$$5x + 10 = 20$$
$$5x = 10$$
$$x = 2$$

Thus we find that the small vessel can hold 2 litres and the big vessel 5litres.

Only numbers

Remember how we did the pencil-pen problem by writing down just the numbers without using letters? Other problems can also be done like this.

For example, we can start doing the vessel problem by first writing like this:

Small	Large	Water
5	2	20
2	3	19

We write next steps like this:

×2	5	2	20	
	2	3	19	×5
	10	4	40	
	10	15	95	

Now we look at only the last two lines and continue like this:

−	10	4	40
	10	15	95
÷ 11	0	11	55
	0	1	5

Thus we get the capacity of the large vessel as 5 litres. Now the capacity of the small vessel can be calculated in the head to be 2 litres. Long before algebra as we now use was developed in Europe, ancient Chinese mathematicians did problems of this kind in this manner.

(1) The price of two kilograms of orange and three kilograms of apple is 520 rupees. The price of three kilograms of orange and two kilograms of apple is 480 rupees. What is the price of each ?

(2) A wire one metre long is cut into two pieces, one of which is bent into a square and the other into an equilateral triangle. Three times the side of the square and two times the side of the equilateral triangle makes 71 centimetres. What are the lengths of the pieces ?

(3) Four years ago, Rahim's age was three times the age of Ramu. After two years, this would become two times. What are their ages now ?

Let's look at another problem:

Four times a number added to three times another number gives 43. Two times the second number subtracted from three times the first gives 11. What are the numbers ?

We can start with taking the first number as x and the second number as y.

Writing the given information as equations, we get

$$4x + 3y = 43 \qquad (1)$$
$$3x - 2y = 11 \qquad (2)$$

As before we multiply Equation (1) by 3 and Equation (2) by 4 to make the multipliers of x the same:

$$(1) \times 3: \; 12x + 9y = 129 \qquad (3)$$
$$(2) \times 4: \; 12x - 8y = 44 \qquad (4)$$

Now subtracting (4) from (3) and simplifying, we get

$$12x + 9y - (12x - 8y) = 129 - 44$$
$$12x + 9y - 12x + 8y = 85$$
$$17y = 85$$
$$y = 5$$

And using $y = 5$ in (1), we can find x:

$$4x + 15 = 43$$
$$4x = 28$$
$$x = 7$$

Thus we see that the first number is 7 and the second, 5

One more problem:

The sum of two numbers is 28 and their difference is 12. What are the numbers ?

Remember doing such a problem before ?

Let's take the larger number as x and smaller number as y and write out the given information as equations:

Sum of the numbers $\quad x + y = 28$

Difference $\quad x - y = 12$

Adding up these equations, we get,

$$(x + y) + (x - y) = 28 + 12$$

That is, $\quad 2x = 40$

$$x = 20$$

What about their difference ?

$$(x + y) - (x - y) = 28 - 12$$

$$x + y - x + y = 16$$

That is, $\quad 2y = 16$

$$y = 8$$

Thus the larger number is 20 and the smaller one is 8.

We also get a general principle from this problem:

The sum and difference of two numbers added together gives twice the larger number:

$$(x + y) + (x - y) = 2x$$

When the difference is subtracted from the sum, we get twice the smaller number:

$$(x + y) - (x - y) = 2y$$

Math and reality

A rectangle of perimeter 10 metres is to be constructed. The length should be 5.5 metres more than the breadth. What should be the length and breadth?

If we take the breadth as x metres the length should be $x + 5.5$ metres. Since the perimeter is to be 10 metres, we should have

$$x + (x + 5.5) = \frac{10}{2} = 5$$

That is,

$$2x + 5.5 = 5$$

which gives

$$2x = -0.5$$

This is not right. How can the length of the side of a rectangle be a negative number ?

This just means that no rectangle can be made satisfying these conditions

Had we taken length as x and breadth as y, we would have got

$$x + y = 5$$

$$y - x = 5.5$$

And we would have immediately seen that there are no positive numbers satisfying these conditions. (The sum of two positive numbers cannot be less than their difference, right?)

(1) The difference of the two smaller angles of a right triangle is $20°$. Calculate all three angles.

(2) When a larger number is divided by a smaller number, the quotient and remainder are both 2. When 5 times the smaller is divided by the larger, the quotient and remainder are still both 2. What are the numbers ?

(3) The sum of the digits of a two digit number is 11. The number got by interchanging the digits is 27 more than the original number. What is the number ?

(4) The price of 17 trophies and 16 medals is 2180 rupees. The price of 16 trophies and 17 medals is 2110 rupees. What is the price of each ?

(5) An object starts with a speed of u m/s and travels along a straight line. If the speed increases at the rate of a m/s every second, the distance travelled in time t seconds is $ut + \frac{1}{2}at^2$. The distance travelled in 2 seconds is 10 metres and the distance travelled in 4 seconds is 28 metres. What was the starting speed ? What is the rate at which speed is increasing ?

(6) A two digit number is equal to 6 times the sum of its digits. The number got by interchanging the digits is 9 more than 4 times the sum of the digits. What is the number ?

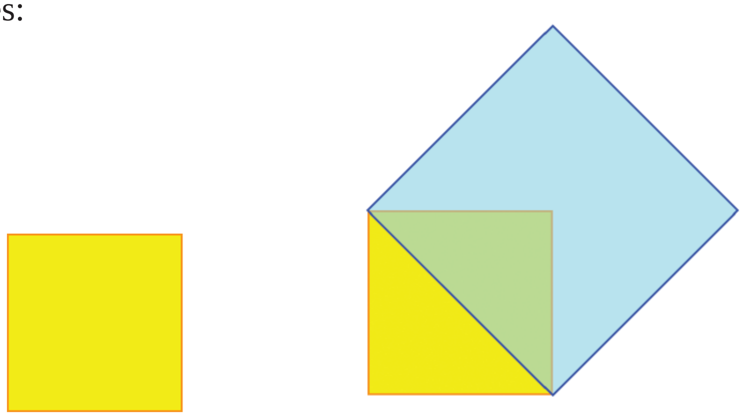
(7) 11 added to a number gives twice another number. 20 added to the second number gives twice the first number. What are the numbers ?

2

New Numbers

Lengths and numbers

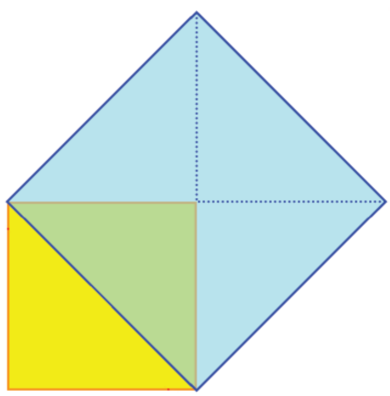
See these pictures:



A square, and on its diagonal another square.

How many times the area of the small square is the area of the large square ?

The diagonal splits the small square into two equal right triangles. How many such right triangles make up the big square ?



So, the area of the big square is twice the area of the small square.

If the side of the small square is 1 metre, then its area is 1 square metre; and the area of the big square is 2 square metre.

What is the length of a side of the big square?

Since it is the diagonal of the small square, it is larger than 1 metre.

Since it is the third side of a triangle of other two sides 1 metre each, it is less than 2 metres.

The length can be a fraction between 1 and 2. But then its square must be 2, being the side of a square of area 2 square metres.

What fraction has its square equal to 2 ?

Can it be one and a half ?

$$\left(1\frac{1}{2}\right)^2 = 1 + 1 + \frac{1}{4} = 2\frac{1}{4}$$

It's a bit too large. How about one and a quarter?

$$\left(1\frac{1}{4}\right)^2 = 1 + \frac{1}{2} + \frac{1}{16} = 1\frac{9}{16}$$

A bit too small now.

What about one and a third?

$$\left(1\frac{1}{3}\right)^2 = 1 + \frac{2}{3} + \frac{1}{9} = 1\frac{7}{9}$$

That too is smaller than what we seek, but better than one and a quarter.

If we go on trying different fractions, the squares get closer and closer to 2, but never exactly 2. In fact, we can prove using algebra that it isn't possible (See the appendix at the end of the lesson).

In other words,

The square of any fraction is not equal to 2

We have a real problem here:

On one hand, if the length of the diagonal of a square of side 1 metre is a fractional multiple of 1 metre, then the square of that fraction must be 2 (We have seen that even if the side of a square is a fraction, its area is the square of the fraction.)

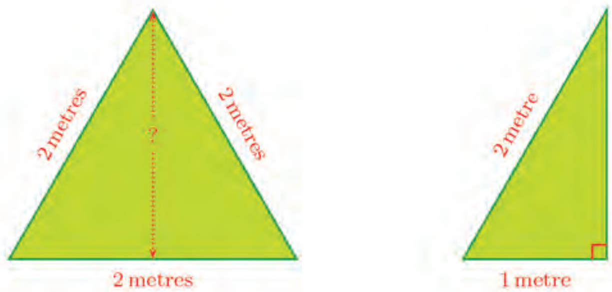
On the other hand, there is no fraction whose square is 2.

Thus we are led to this conclusion:

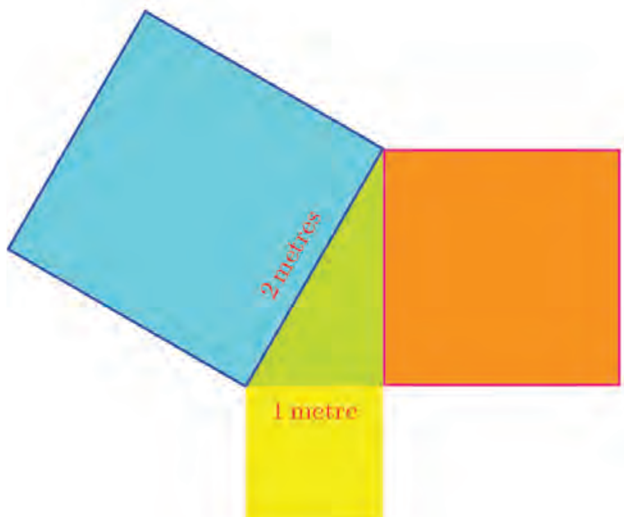
The length of the diagonal of a square of side 1 unit cannot be expressed as a fraction.

There are other lengths like this, which cannot be expressed as natural numbers or fractions. For example, let's compute the height of an equilateral triangle of sides 2 metres.

If we cut such a triangle into two along its height and take one piece, we get a right triangle:



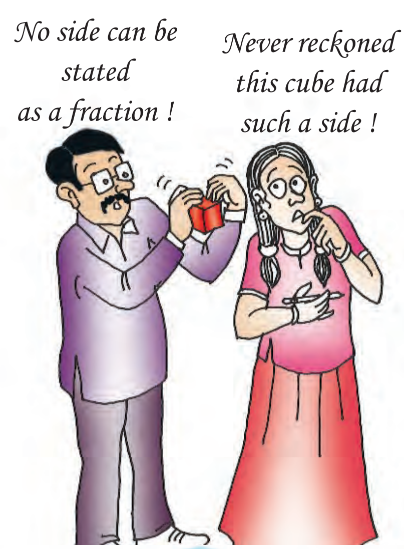
To compute the length of its third side, let's draw squares on all sides



By the Pythagoras Theorem, the area of the orange square is $4 - 1 = 3$ square metres. So, if its side is a fractional multiple of 1 metre, then the square of this fraction must be 3.

Just as we show that the square of no fraction is 2, we can also show that the square of no fraction is 3. So, the height of this triangle cannot be expressed as a fraction.

Let's look at another example. Suppose we want to make a cube of volume 2 cubic centimeters. What should be the length of a side? Just as the square of no fraction is equal to 2, the cube of no fraction is equal to 2 either. So, we cannot express the length of a side of this cube as a fraction.



In many such contexts, we need lengths which cannot be expressed as fractions.

Number evolution

Convert everything to numbers and try to make sense of the world through such numbers and their interrelations, this is one of the primary concerns of mathematics.

Depending on the nature of the things measured, different kinds of numbers have to be invented. During the period when humans were just hunter gatherers, they needed only such numbers as the number of people in a group or the number of cattle they owned. At that time only natural numbers was necessary.

Around BCE 5000 they began to settle down by the side of the great rivers and started extensive agriculture. Then they had to measure lengths and areas to mark farmlands and to build houses. Fractions were invented at this stage. Fractions are also necessary for fair division. New numbers became necessary with the realization that not all measurements can be indicated by fractions.

Later, new numbers were created not only for physical needs, but also as mathematical conveniences. Negative numbers and complex numbers were invented for such a purpose. That such numbers also were found to be useful in physical sciences is another side of the story.

Measures and numbers

To indicate lengths which cannot be expressed as natural numbers or fractions, we have to create new numbers. Let's have a look at our first example. How do we represent the length of the diagonal of a square of side 1 metre ?

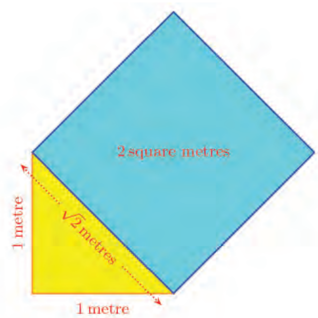
Let's put it this way: how do we indicate the side of a square of area 2 square metres?

If the side of a square is a natural number or fraction, the length of a side is the square root of the area.

For example, the side of a square of area 4 square metres is $\sqrt{4} = 2$ metres; if the area is $2\frac{1}{4}$ square metres, the length of a side is $\sqrt{2\frac{1}{4}} = 1\frac{1}{2}$ metres.

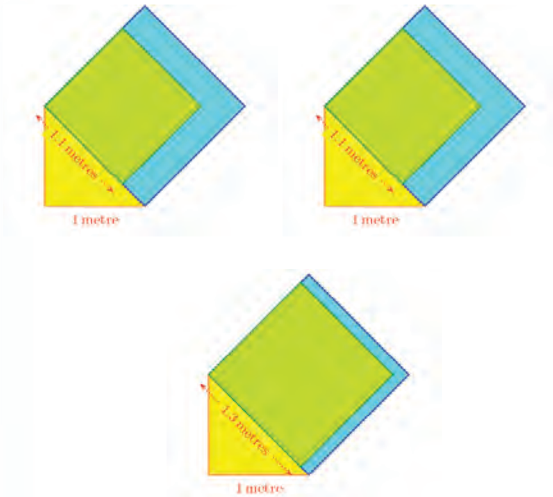
In the same way, we write

The side of a square of area 2 square metres is $\sqrt{2}$ metres



Just giving a symbol to denote a length isn't enough; to know its size we must be able to compare it with known lengths.

To do it, we must find lengths which can be expressed as fractions and which get closer and closer to this length. If such lengths are marked on the diagonal itself, the squares on these lengths get closer and closer to the square on the diagonal.



In terms of just numbers, this means the squares of the fractions giving these lengths get nearer and nearer to 2. We can find these squares using a calculator:

$$1.1^2 = 1.21$$

$$1.2^2 = 1.44$$

$$1.3^2 = 1.69$$

$$1.4^2 = 1.96$$

$$\mathbf{1.5^2 = 2.25}$$

What do we see here?

$$1.4^2 < 2 < 1.5^2$$

Let's explain this method of finding fractions whose squares get closer and closer to 2.

Considering only natural numbers, we get,

$$1^2 < 2 < 2^2$$

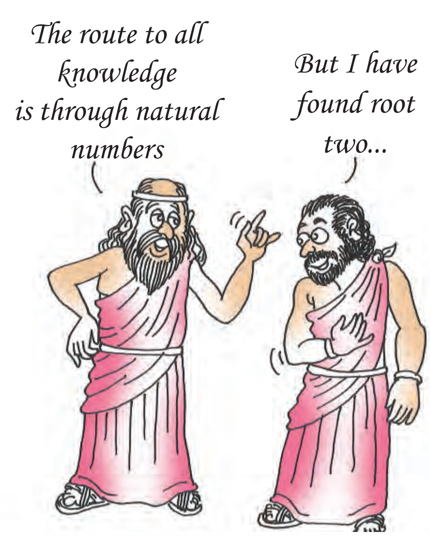
Collapsing beliefs

Pythagoras, who lived around the sixth century BCE, and his followers believed that all measurements could be compared using just natural numbers. More precisely, that any two measurements can be represented together as a ratio of natural numbers.

But the ratio of the lengths of the diagonal and side of a square cannot be thus represented. For if this ratio can be given as $a : b$, where a and b are natural numbers, then the diagonal must be $\frac{a}{b}$ times the side, which means the square of the length of the diagonal must be $\left(\frac{a}{b}\right)^2$ times the square of the length of the side; since the area of the square on the diagonal is twice the area of the square on the side, this would mean, $\left(\frac{a}{b}\right)^2 = 2$, which we have seen is impossible.

It is believed that this was first discovered by Hippasus, himself a disciple of Pythagoras.

Pairs of lengths, such as the diagonal and side of a square, which cannot be compared using a ratio of natural numbers are called incommensurable magnitudes.



Suppose we take tenths also into our reckoning

$$\left(1\frac{4}{10}\right)^2 < 2 < \left(1\frac{5}{10}\right)^2$$

What if we take hundredths also?

$$1.41^2 = 1.9881$$

$$1.42^2 = 2.0164$$

Thus

$$1.41^2 < 2 < 1.42^2$$

or in other words

$$\left(1\frac{41}{100}\right)^2 < 2 < \left(1\frac{42}{100}\right)^2$$

Continuing like this, we can see

$$1.4^2 = 1.96 \qquad 1.5^2 = 2.25$$

$$1.41^2 = 1.9881 \qquad 1.42^2 = 2.0164$$

$$1.414^2 = 1.999396 \qquad 1.415^2 = 2.002225$$

$$1.4142^2 = 1.99996164 \qquad 1.4143^2 = 2.00024449$$

$$1.41421^2 = 1.9999899241 \qquad 1.41422^2 = 2.0000182084$$

That is if we take up to the fifth decimal place (up to hundred thousandths),

$$1.41421^2 < 2 < 1.41422^2$$

In short,

The squares of the fractions $1\frac{4}{10}$, $1\frac{41}{100}$, $1\frac{414}{1000}$, $1\frac{4142}{10000}$, $1\frac{41421}{100000}$ and so on get nearer and nearer to 2.

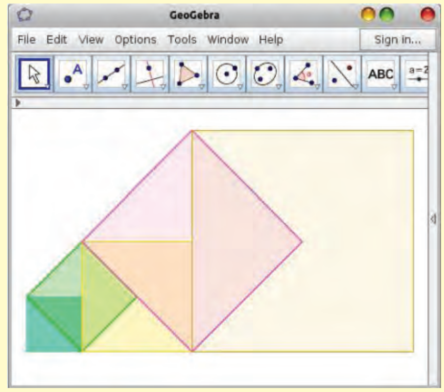
In terms of decimals,

The squares of the fractions 1.4, 1.41, 1.414, 1.4142, 1.41421 and so on get nearer and nearer to 2.

This we write in shortened form like this:

$$\sqrt{2} = 1.41421...$$

We also say that the number $\sqrt{2}$ is 1.4 up to one decimal place, 1.41 up to two decimal places and so on.



In the picture, the side of the smallest square is 1 centimetre. Calculate the area and the side of the largest square. Draw this in GeoGebra (Use the Regular Polygon tool). Use Area tool to find the area of each square. The sides of which of these squares can be expressed as fractions ?

Nearer and Nearer

$$2 - 1.4^2 = 0.04$$

$$2 - 1.41^2 = 0.0119$$

$$2 - 1.414^2 = 0.000604$$

$$2 - 1.4142^2 = 0.00003836$$

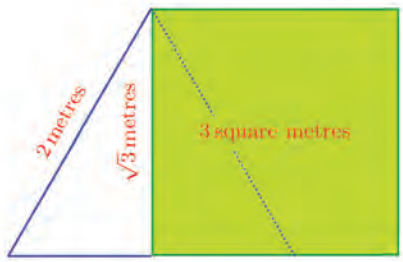
$$2 - 1.41421^2 = 0.0000100759$$

These are written,

$$\sqrt{2} \approx 1.4$$
$$\sqrt{2} \approx 1.41$$

Here the symbol ≈ is to be read "approximately equal to".

Similarly, since the area of the square drawn on the height of an equilateral triangle of side 2 metres is 3 square meters (as we have seen earlier), we can denote this height by $\sqrt{3}$



By computations as done earlier, we can find that the squares of the fractions 1.7, 1.73, 1.732 and so on get nearer and nearer to 3. This also we shorten as

$$\sqrt{3} = 1.73205...$$

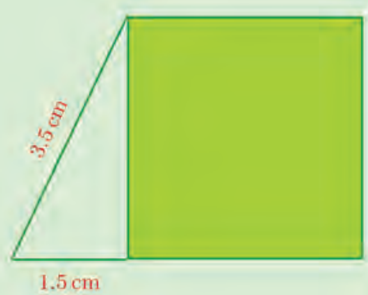
Generally speaking,

$\sqrt{x}$ is the side of a square of area x, for any positive number x

In some cases, $\sqrt{x}$ may be a natural number or a fraction. If not, we can compute fractions whose squares get nearer and nearer to x and write $\sqrt{x}$ in decimal form.

(1) We have seen in class 8 that any odd number can be written as the difference of two perfect squares. Use this to draw squares of area 7 square centimetres and 11 square centimetres. What are the lengths of the sides of these squares?

(2) What is the area of the square in the picture? What is the length of its sides?



Line and root

For any number x, we have

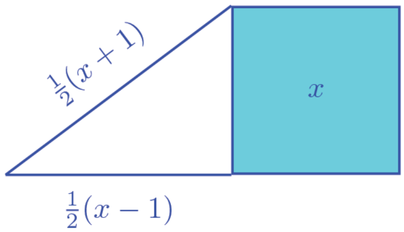
$$(x+1)^2 - (x-1)^2 = 4x$$

(The lesson **Square Equations** in Class 8) This we can rewrite as

$$x = \left(\tfrac{1}{2}(x+1)\right)^2 - \left(\tfrac{1}{2}(x-1)\right)^2$$

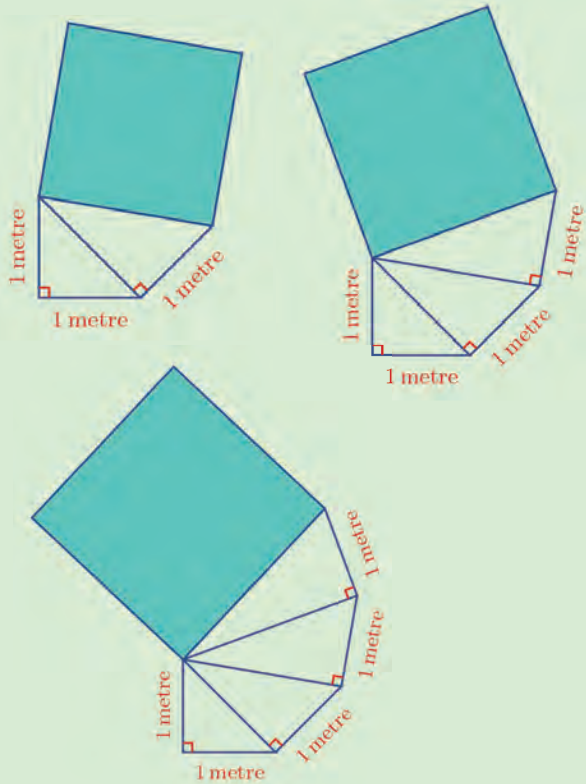
and use this to draw a square of area x for any x.

For $x > 1$, this is how we do it:



For $x < 1$, we take the length of the base of triangle as $\frac{1}{2}(1-x)$

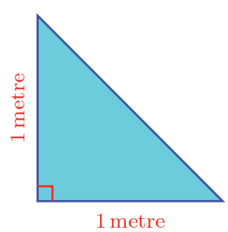
(3) Calculate the area of the square and the length of its sides in each of the pictures below:



(4) Find three fractions greater than $\sqrt{2}$ and less than $\sqrt{3}$

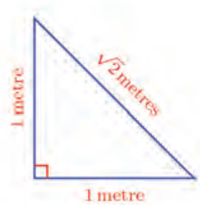
Addition and subtraction

What is the area of a right triangle of perpendicular sides 1 metre each ?



And the perimeter?

We know that its hypotenuse is $\sqrt{2}$ metres



So to calculate the perimeter, we have to add 2 metres and $\sqrt{2}$ metres.

We write this length as $2+\sqrt{2}$.

The fractions 1.4, 1.41, 1.414, 1.4142 and so on are approximate values of $\sqrt{2}$.

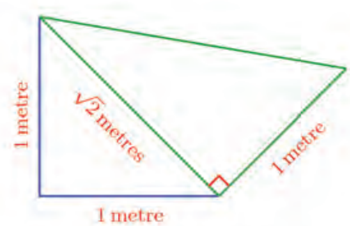
So, by adding 2 to each of these, we get fractions approximating $2+\sqrt{2}$; that is, the fractions, 3.4, 3.41, 3.414, 3.4142 and so on.

This we write as

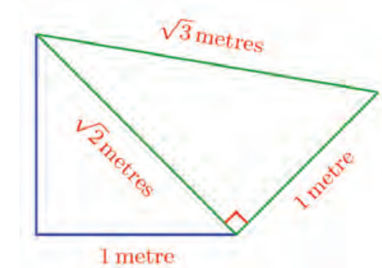
$$2+\sqrt{2} = 3.4142...$$

If we decide to have accuracy only up to a centimetre, we can take the perimeter as 3.41 metres; instead if we need accuracy up to a millimetre, we take the perimeter as 3.414 metres.

Now suppose we draw another triangle with the hypotenuse of the first as its base, as below:



We've seen that the third side of this triangle is $\sqrt{3}$ metres.



Decimal forms

Decimals were first used as a shorthand for fractions with powers of 10 as denominators, such as

$$\frac{7}{10} = 0.7, \qquad \frac{21}{100} = 0.21$$

$$\frac{1}{2} = \frac{5}{10} = 0.5, \qquad \frac{1}{4} = \frac{25}{100} = 0.25$$

Later this was extended to other fractions, in a different form. For example, since the fractions

$$\frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \dots$$

get closer and closer to $\frac{1}{3}$, we write

$$\frac{1}{3} = 0.333...$$

Similarly, we can compute

$$\frac{1}{6} = 0.1666...$$

$$\frac{1}{11} = 0.090909...$$

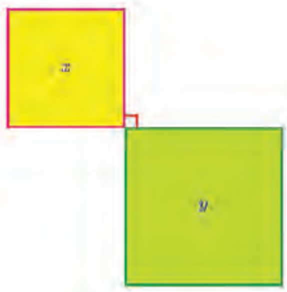
In the same way, we write

$$\sqrt{2} = 1.41421...$$

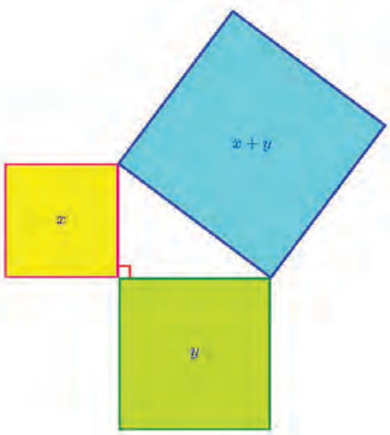
But there's a difference. In the decimal forms of fractions such as $\frac{1}{3}$, $\frac{1}{6}$ or $\frac{1}{11}$ we see groups of digits repeating again and again. In the decimal forms of numbers like $\sqrt{2}$ or $\sqrt{3}$, there are no such repetitions.

Sum and root

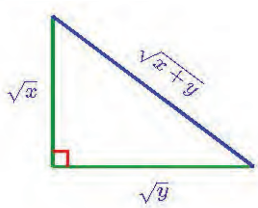
Place two squares of areas x and y like this:



Next let's join the top corrners and draw a square on it:



What are the lengths of the sides of triangle in the middle?



What do get from this?

$$\sqrt{x}+\sqrt{y}>\sqrt{x+y}$$

So the perimeter of this triangle is $1+\sqrt{2}+\sqrt{3}$. To get factions approximating $\sqrt{2}+\sqrt{3}$, we must add the approximations to each of these in order:

1.4	1.41	1.414	...	$\rightarrow$	$\sqrt{2}$
1.7	1.73	1.732	...	$\rightarrow$	$\sqrt{3}$
3.1	3.14	3.146	...	$\rightarrow$	$\sqrt{2}+\sqrt{3}$

Adding 1 to each of these give approximations to $1+\sqrt{2}+\sqrt{3}$

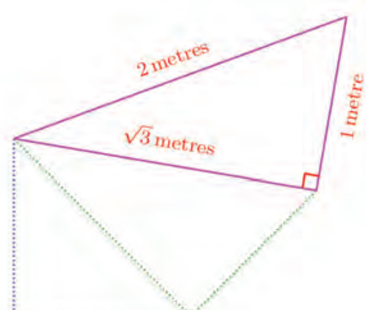
Thus the perimeter of the new triangle is 4.146 metres, correct to a millimetre

How much larger is the perimeter of this triangle than that of the first one ?

We can say approximately $4.146 - 3.414 = 0.732$ metres. Or we can compute like this:

$$\begin{aligned}(1+\sqrt{2}+\sqrt{3}) - (2+\sqrt{2}) &= 1+\sqrt{3}-2\\ &= \sqrt{3}-1\\ &\approx 0.732\end{aligned}$$

Now suppose we draw yet another triangle on top of this as below: What are the lengths of the sides ?



How much larger is the perimeter of this than that of the second one?

The perimeter of the new triangle is

$$2 + 1 + \sqrt{3} = 3 + \sqrt{3}$$

Let's compute how much larger is the perimeter, without actually calculating its approximate values.

What is the perimeter of the second triangle? So, the difference in perimeters is

$$(3 + \sqrt{3}) - (1 + \sqrt{2} + \sqrt{3}) = 2 - \sqrt{2}$$

We can calculate this upto three decimals as

$$2 - 1.414 = 0.586$$

That is, the perimeter is about 586 millimetres or 58.6 centimetres more

Subtraction

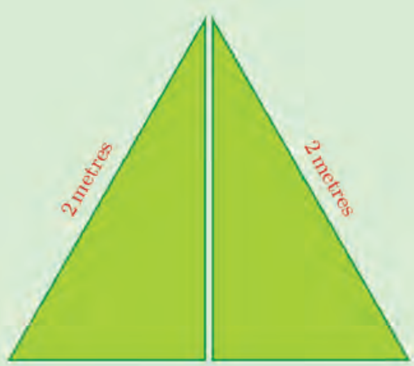
We calculate fractions approximating $\sqrt{3} - \sqrt{2}$, just as we did for $\sqrt{3} + \sqrt{2}$.

1.7	1.73	1.732	...	$\rightarrow$	$\sqrt{3}$
1.4	1.41	1.414	...	$\rightarrow$	$\sqrt{2}$
0.3	0.32	0.318	...	$\rightarrow$	$\sqrt{3} - \sqrt{2}$

$$\sqrt{3} - \sqrt{2} \approx 0.318$$

(1) The hypotenuse of a right triangle is $1\frac{1}{2}$ metres and one of the other sides is $\frac{1}{2}$ metre. Calculate its perimeter, up to a centimetre.

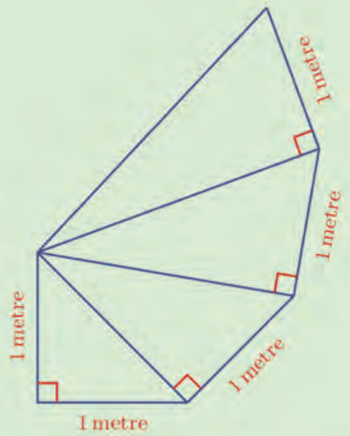
(2) The picture below shows an equilateral triangle cut into two triangles along a line through the middle:

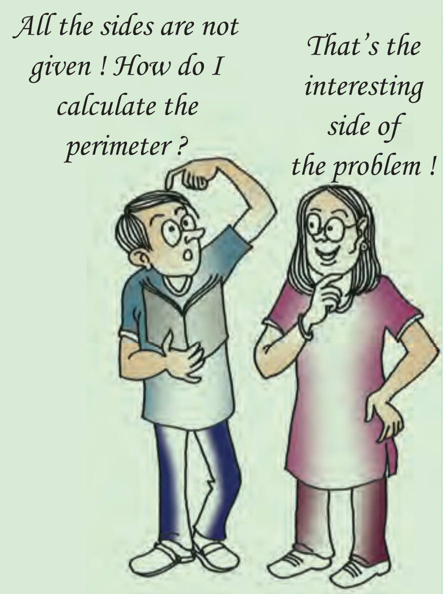


(i) What is the perimeter of one of these ?

(ii) How much is it less than the perimeter of the whole triangle ?

(3) We've seen how we can go on drawing right triangles like this:





(i) What are the lengths of the sides of the tenth triangle in this pattern?

(ii) How much more is the perimeter of the tenth triangle than that of the ninth?

(4) What is the hypotenuse of a right triangle with perpendicular sides $\sqrt{2}$ centimetres and $\sqrt{3}$ centimetres ? How much more is the sum of the perpendicular sides than the hypotenuse ?

Appendix

Let's see how we can prove that there is no fraction whose square is 2.

We start by looking at the specialities that the numerator and denominator of such a fraction. We know that every fraction has various forms and among these there's one in its lowest terms, that is, a form in which the numerator and denominator have no common factor. Let's take this form of the fraction we are seeking, the one whose square is 2, as $\frac{x}{y}$. So, x and y are natural numbers with no common factor.

What other properties do they have?

$$\left(\frac{x}{y}\right)^2 = 2$$

That is,

$$\frac{x^2}{y^2} = 2$$

which gives

$$x^2 = 2y^2$$

So, x^2 is an even number. What about x itself ?

The squares of odd numbers are all odd numbers (and the squares of even numbers, even). Since x^2 is even, so must be x itself.

Now x is an even number and x and y have no common factors. So, y cannot be an even number (2 is a common factor of any two even numbers, right ?). This means y is an odd number. Thus in the fraction we are seeking, the numerator is even and the denominator is odd. Can we find anything more ? Let's continue our investigation.

Since x is an even number, we can divide it by 2 to get a natural number. That is $\frac{x}{2}$ is a natural number. Let's write it as z:

$$\frac{x}{2} = z$$

That is,

$$x = 2z$$

Now we can write $2z$ in the place of x in our first equation $x^2 = 2y^2$:

$$(2z)^2 = 2y^2$$

That is

$$4z^2 = 2y^2$$

From this we get

$$y^2 = 2z^2$$

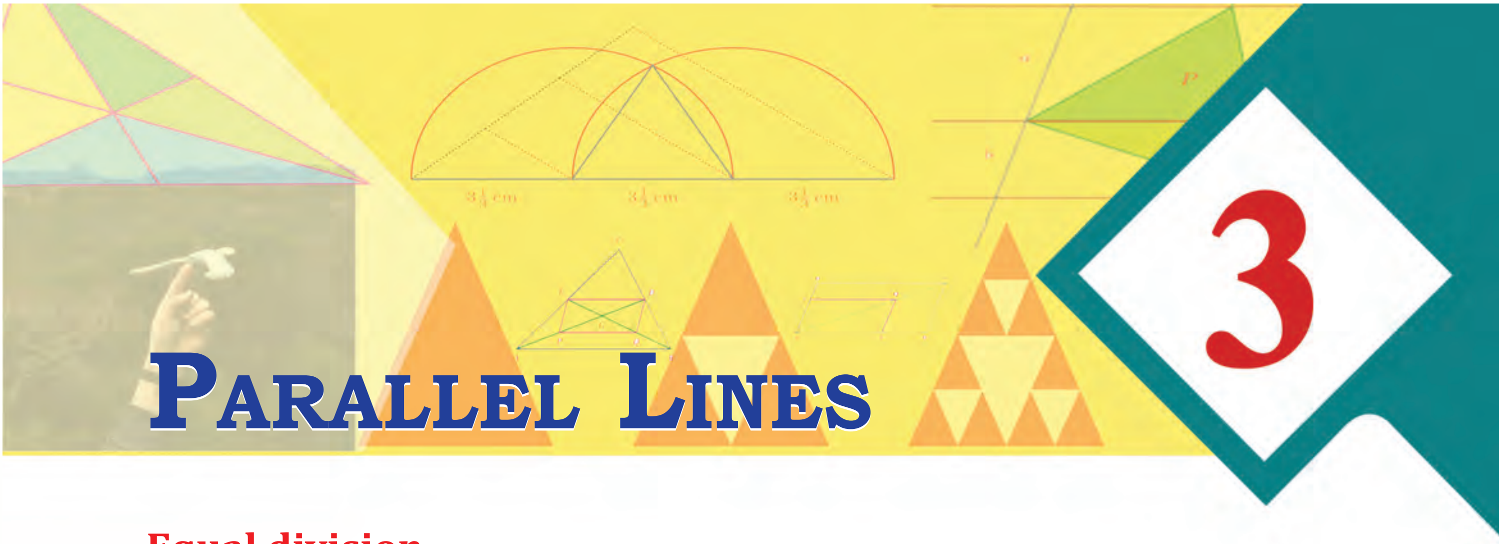
This means y^2 is an even number. From this we can see, as before in the case of x, that y itself is even.

But how can this be ? We've already noted that y is an odd number.

What happened here ?

For a fraction in the lowest terms whose square is 2, we first saw that the denominator is an odd number; further analysis showed the denominator must be an even number. And we cannot have both.

Thus we see there is no fraction whose square is 2.



3 Parallel Lines

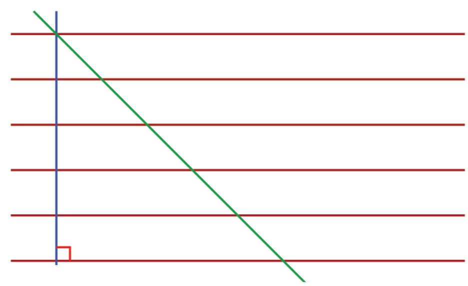
Equal division

We've learnt many things about parallel lines, and drew many a figure using these. There's much more.

See this picture:

A bunch of parallel lines and a line perpendicular to them all.

The parallel lines are drawn the same distance apart. In other words, the parallel lines cuts the perpendicular into equal pieces.



What if we draw a slightly slanted line instead of a perpendicular ?

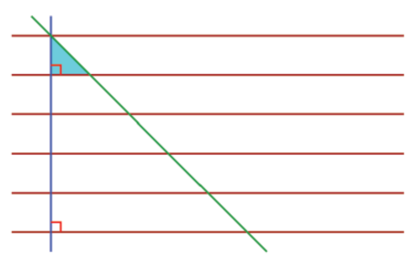
Doesn't it look as if the parallel lines cut

this also into equal pieces? How do we check?

We can measure them;

The fun in math lies in finding things by reasoning than by actually doing.

Do you see a small right triangle at the top of the picture?



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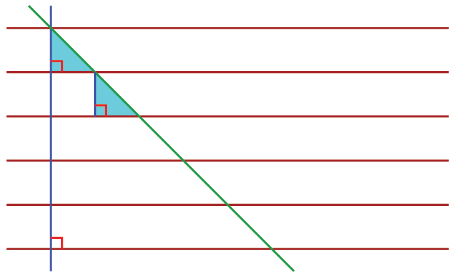
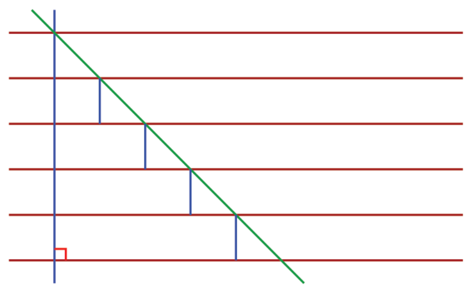
Its vertical side is a piece of the perpendicular; and the hypotenuse is a piece of the slanted line.

If we draw one more small perpendicular, we get another triangle just below the first:



The vertical (blue) sides of these triangles are equal (why?) Also, since these perpendiculars are parallel, the slanted (green) line is equally inclined to both.

In other words, the vertical sides of these triangles are equal and so are its angles on either side of these lines. So, their hypotenuses must also be equal, right?



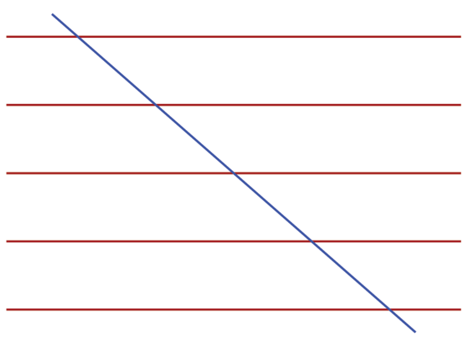
Thus the top two pieces of the slanted line are of equal length.

In the same way, we can see that the other pieces are all equal.

We state this as a general result:

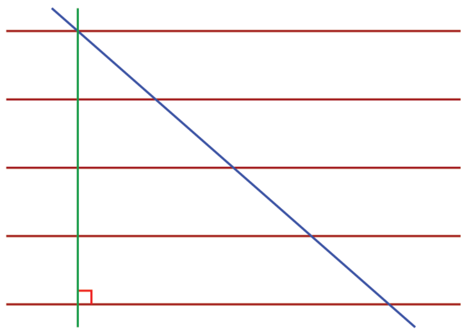
Parallel lines the same distance apart cut any other line into equal pieces.

What about the other way round? That is, suppose that some parallel lines cut a slanted line into equal pieces. Can we say that they are the same distance apart ?

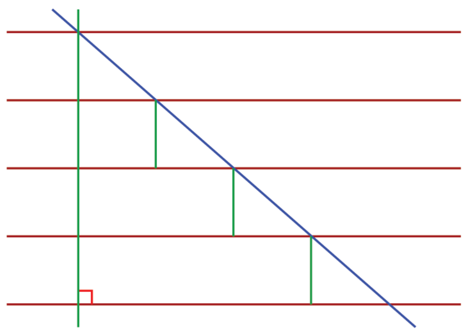


The parallel lines in the picture cut the slanted blue line into equal pieces. The question is whether the distances between adjacent pairs the same.

In other words, we must check whether the parallel lines cut the green perpendicular also into equal pieces.



Let's draw small perpendiculars as before:

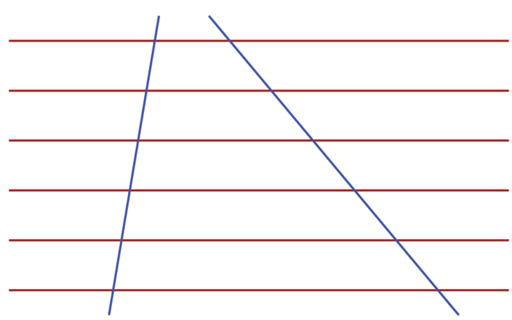


As in the first case, all the little triangles have the same angles. And their hypotenuses are also equal. So, their vertical sides are also equal.

Thus the parallel lines are the same distance apart.

We state this also as a general result:

Parallel lines which cut some line into equal pieces are equal distance apart.



Using these two results, we can see another thing. Suppose a bunch of parallel lines cut some line into equal pieces. By the result just seen, they are equal distance apart.

So, by the first result, they cut any other line also into equal pieces.

Thus we have the result below:

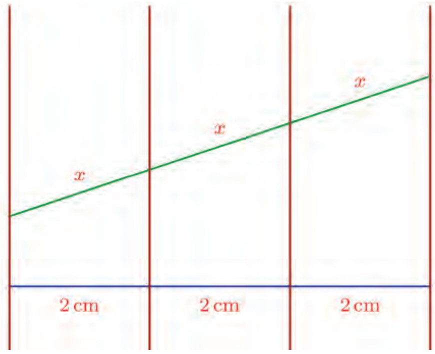
Parallel lines which cut one line into equal pieces, cut all lines into equal pieces

We can use this to cut any line into any number of equal pieces.

For example, let's see how we can divide a 7 centimetre long line into three equal pieces.

This is easy for a 6 centimetre long line.

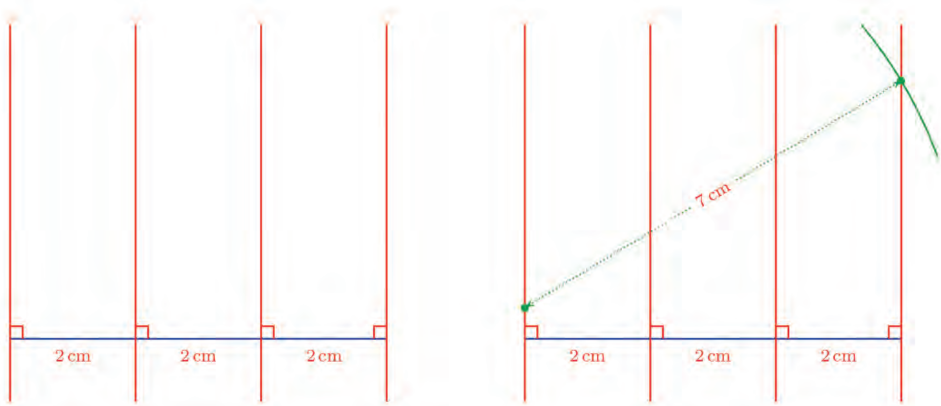
Four parallel lines which divide a 6 centimetre long line into three equal parts, divide any other line into three equal pieces, don't they?



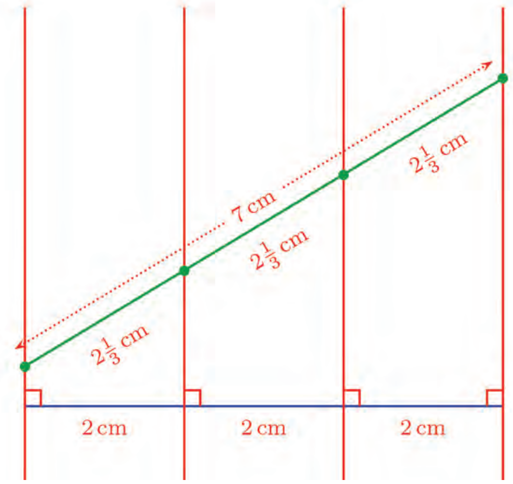
If we make the other line 7 centimetre long ?

Now the way is clear to solve our problem.

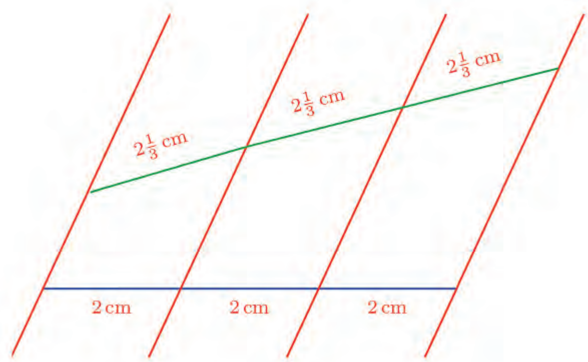
Draw a line 6 centimetre long and draw on it, perpendiculars 2 centimetres apart. From any point on the first perpendicular, draw an arc of radius 7 centimetres and mark the point where it cuts the last perpendicular:



Joining these two points, we get a 7 centimetre long line cut into three equal pieces.



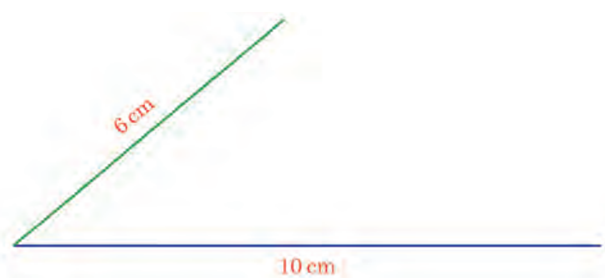
It is not really necessary that we should draw perpendiculars to the first line. Slanted lines also will do; but they must be parallel:



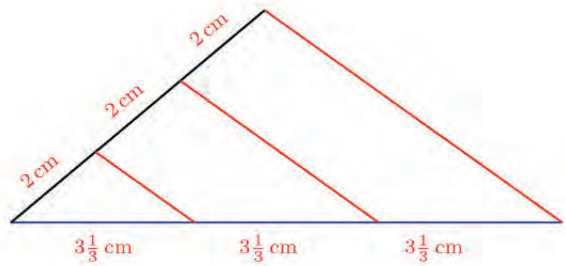
Another problem:

Draw a line 10 centimetre long and divide it into three equal pieces.

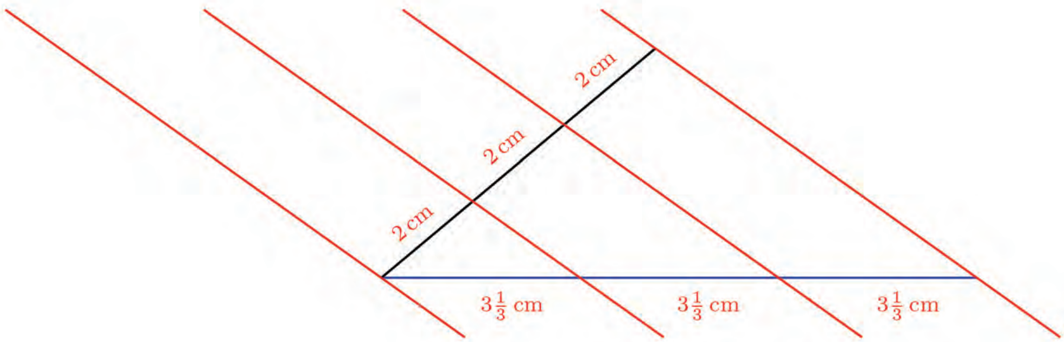
It is more convenient to draw horizontally the line to be divided. Then we can draw the line which helps the cutting, slanted from one end of the first:



Now draw a line joining the other ends of the two lines. Next we need only divide the top line into three equal parts and draw parallels through these points:

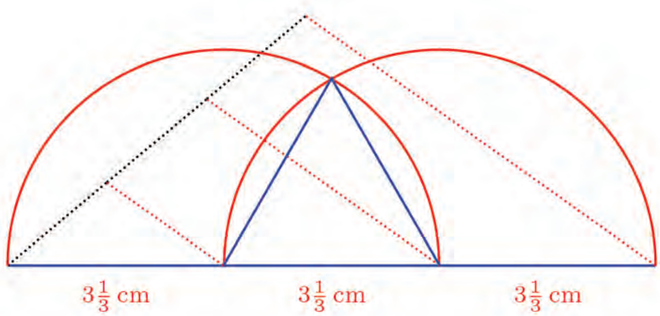


If this is not clear, extend these parallel line a little and draw a fourth parallel:



Thus the 10 centimetre long line is cut into three equal parts.

How about drawing a triangle joining these pieces?



This is an equilateral triangle, since all three sides are of the same length. And their sum is 10 centimetres. That is, the perimeter of the triangle is 10 centimetres.

Now suppose we want to divide a 10 centimetre long line into four equal parts instead of three equal parts. What is a convenient length for the slanted line at the top?

And after doing such a division, we can also draw a square of perimeter 10 centimetres. Try it.

And also these:

(1) Draw an equilateral triangle of perimeter 11 centimetres.

(2) Draw a square of perimeter 15 centimetres.

(3) Draw a regular hexagon of perimeter 20 centimeters.

Unequal division

See this picture:

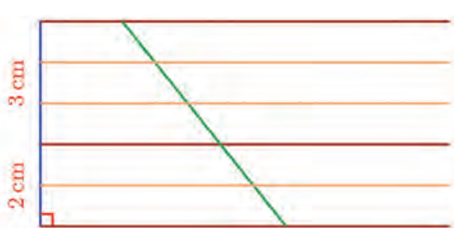
Three parallel lines with different distances between them

Let's draw a slanted line as before:

We can tell at a glance the lengths of the pieces of the green line are not 2 centimetres and 3 centimetres.

Do these lengths have any relation with 2 and 3 ?

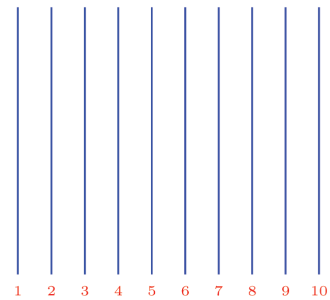
Suppose we draw three more parallel lines to make six equally spaced parallel lines ?



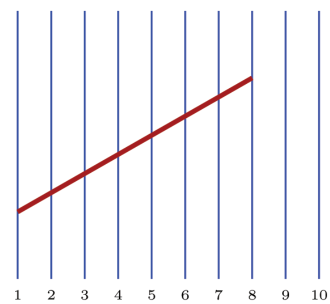
Now the slanted line is split into five equal parts. The top three pieces make up the longer of the two pieces we saw first; and the bottom two make up the shorter piece.

Parallel division

Draw some equally spaced parallel lines on a sheet of paper:

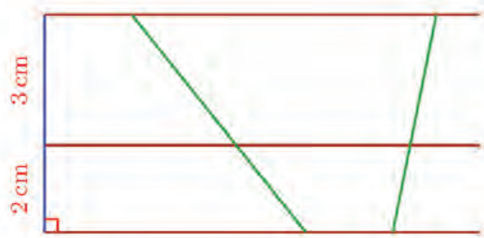


Now to divide a thin wire into seven equal pieces, all we need to do is to place it across eight of these line as shown below:



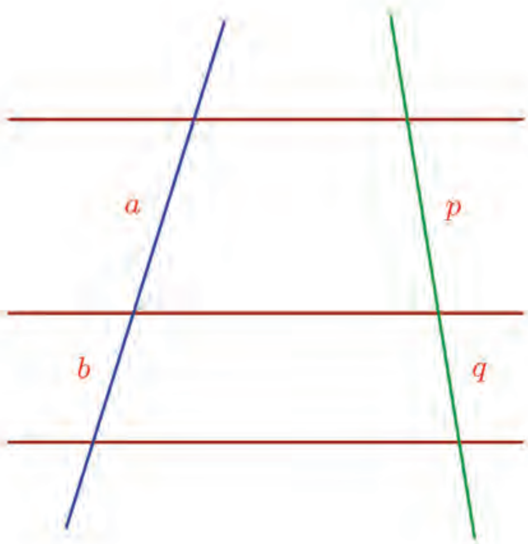
In other words, the shorter and longer parts of the slanted line are in the ratio 2 : 3.

What if we draw another line ?



The actual lengths of the parts are different, but still two of the equal parts make up the shorter part and three of them, the longer part. That is, the ratio is still 2 : 3

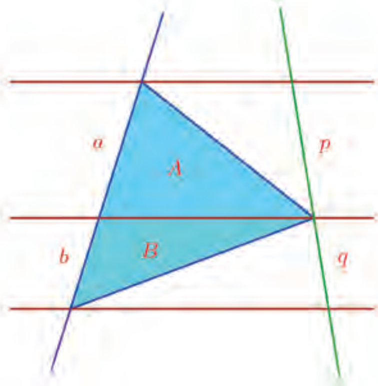
Do any three parallel lines divide all other lines in the same ratio?



In the picture above, there are three horizontal lines which are parallel and two slanted lines which cut through them. Let's denote the lengths of the pieces of the left line as a, b and those of the right line as p, q.

We want to check whether the ratios $a : b$ and $p : q$ are the same. In other words, the question is whether $\frac{a}{b} = \frac{p}{q}$.

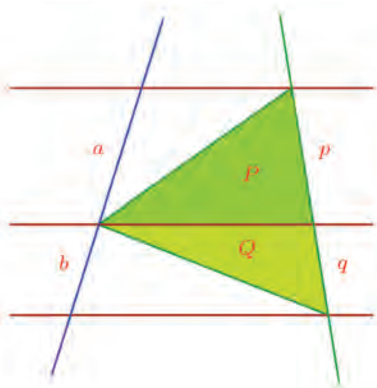
To check this, we first change the ratio of lengths to ratio of areas:



We know that the ratio of the areas of the top and bottom blue triangles is also $a : b$. That is, if the areas of these triangles are denoted as A and B, then

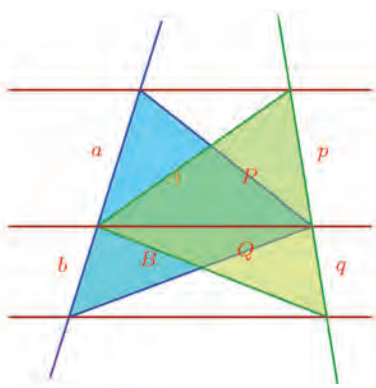
$$\frac{a}{b} = \frac{A}{B}$$

Similarly, we can also convert the ratio of the lengths p and q also to a ratio of areas:

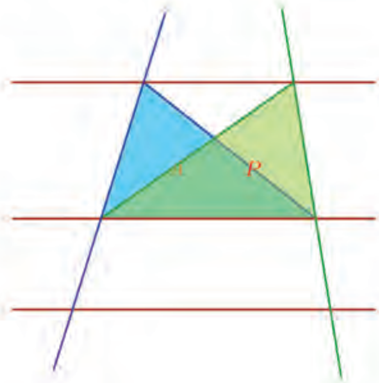


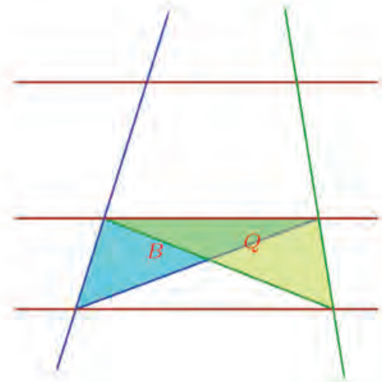
If we denote the areas of the green triangles as P and Q as in the picture above, $\frac{p}{q} = \frac{P}{Q}$

Now let's put all the triangles together:



There is a blue triangle and green triangle each at the top and bottom. Let's look at them pair by pair:





Both triangles at the top have the same bottom side; and their top corners are on a line parallel to this side. So, they have the same area.

$$A = P$$

Same is the case with the blue and green triangles at the bottom.

$$B = Q$$

We have already seen that

$$\frac{a}{b} = \frac{A}{B} \text{ and } \frac{p}{q} = \frac{P}{Q}$$

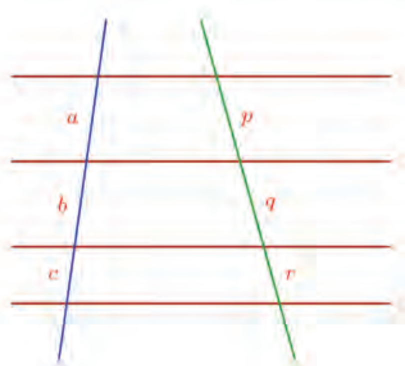
And now we see that $A = P$ and $B = Q$. Thus,

$$\frac{a}{b} = \frac{p}{q}$$

In other words, the three parallel lines divide the left and right lines in the same ratio.

What if there are more than three lines ?

See this picture:



If we just look at the top three lines only, we get $\frac{a}{b} = \frac{p}{q}$, as we did just now.

Instead if we look at the bottom three lines only, we get $\frac{b}{c} = \frac{q}{r}$.

From $\frac{a}{b} = \frac{p}{q}$ and $\frac{b}{c} = \frac{q}{r}$, we get $\frac{a}{b} \times \frac{b}{c} = \frac{p}{q} \times \frac{q}{r}$. That is, $\frac{a}{c} = \frac{p}{r}$.

So among the lengths a, b, c whatever part or times of one is another, the same part or times is the corresponding pieces of p, q, r

Thus the ratio between the lengths a, b, c is same as the ratio between the lengths p, q, r.

We can continue like this for any number of parallel lines.

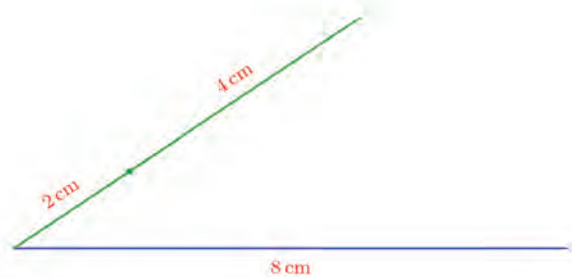
Three or more parallel lines cut any two lines in the same ratio.

We can use this to divide a line in any ratio.

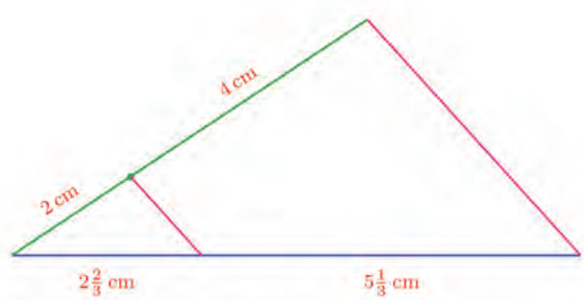
For example, let's see how we divide an 8 centimetre long line in the ratio 1 : 2.

Isn't it easy to divide a 6 centimetre line in this ratio ?

So let's first draw two lines as we did earlier:



Now we need only join the other ends of the lines and draw the parallel line through the point dividing the green line:



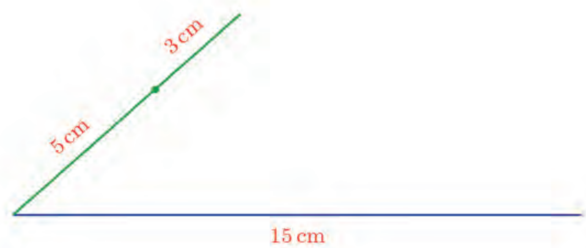
Another problem:

Draw a rectangle of perimeter 30 centimetres and sides in the ratio 5 : 3.

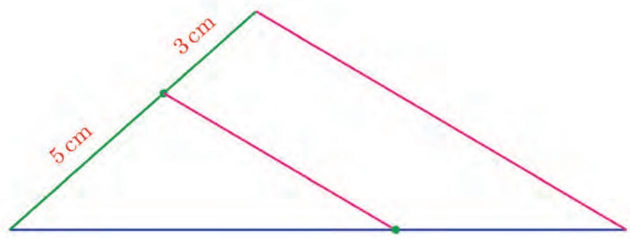
Since the perimeter is 30 centimetres, the sum of the lengths of the sides is 15 centimetres.

So, the length and breadth of the rectangle are the pieces got by dividing a 15 centimetre long line in the ratio 5 : 3.

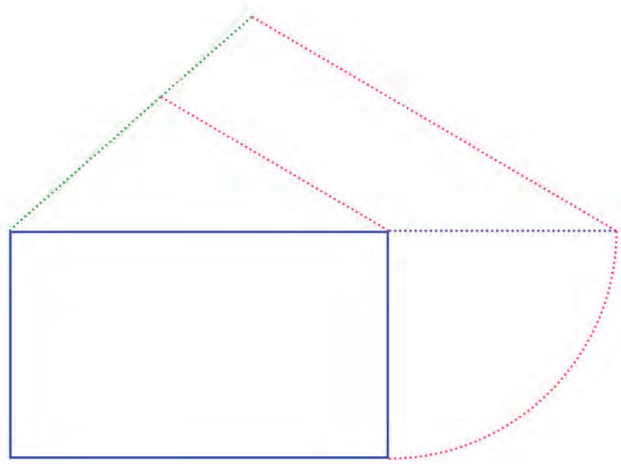
To get these pieces, we start like this:



Then join the other ends of these lines and draw the parallel line:



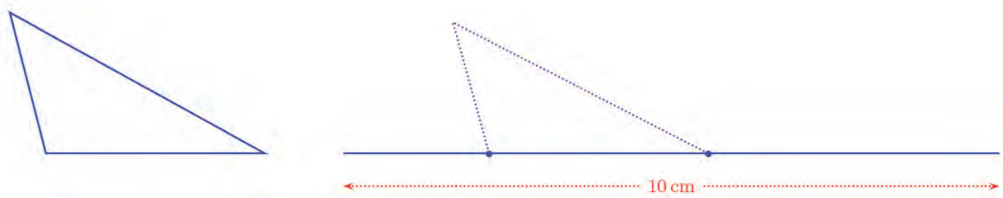
The rectangle we want is the one with the pieces of the bottom (blue) line as width and height:



One more problem:

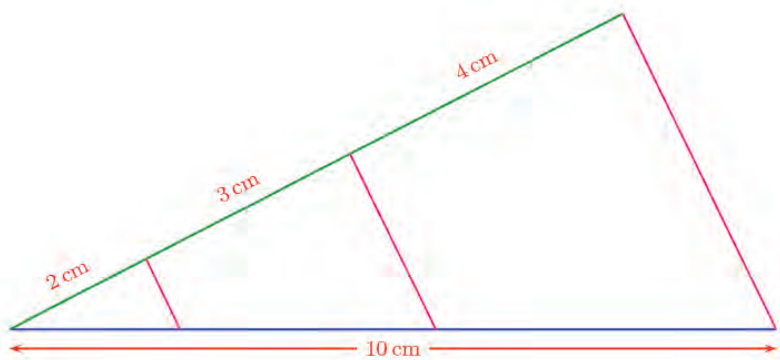
Draw a triangle of perimeter 10 centimetres and the ratio of the sides 2 : 3 : 4

If we unfold the sides of the triangle and spread them along a single straight line, it would be 10 centimetre long:

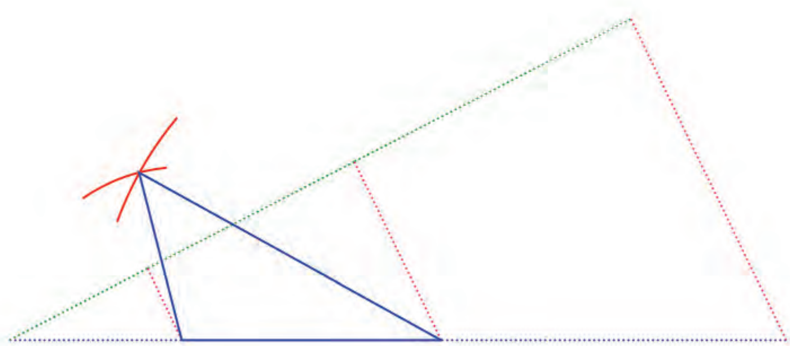


So, to draw the triangle, all we need to do is to first draw a line 10 centimetre long, then divide it in the ratio 2 : 3 : 4 and finally fold the pieces at the end.

What is the length which can be easily divided in this ratio?



Now we can draw the triangle:



(1) Draw a rectangle of perimeter 15 centimetres and sides in the ratio 3: 4.

(2) Draw a triangle of each of the types below, of perimeter 13 centimetres:

   (i) Equilateral

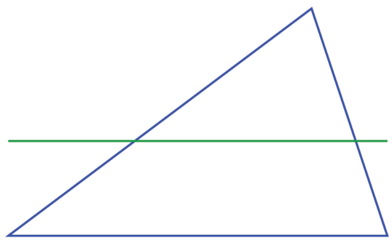
   (ii) Sides in the ratio 3 : 4 : 5

   (iii) Isosceles with lateral sides one and a half times the base

(3) Prove that in any trapezium, the diagonals cut each other in the same ratio.

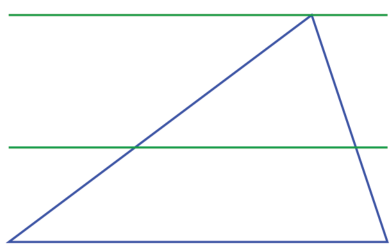
Triangle division

Draw a triangle and a line within it, parallel to one side:



Is there any relation between the pieces into which this line cuts the other sides ?

Suppose we draw another line parallel to the bottom side through the top vertex:

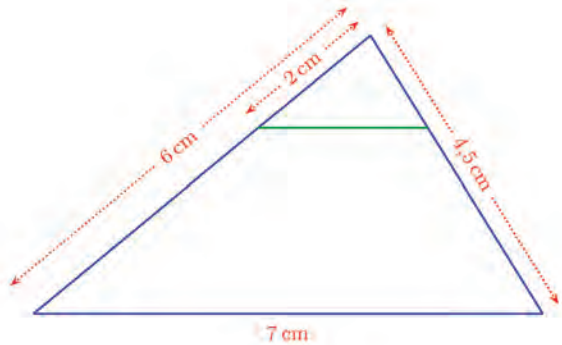


Now there are three parallel lines cutting the left and right sides of the triangle. The ratio of the pieces on either side must be the same, right?

What do we see here?

In any triangle, a line parallel to one of the sides cuts the other two sides in the same ratio.

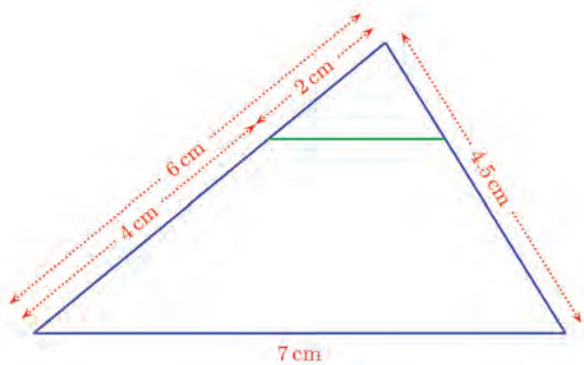
Let’s look at a problem:



In this picture, the green line within the blue triangle is parallel to the bottom side. We want to calculate the ratio in which this line divides the right side.

It divides both left and right sides in the same ratio.

What’s the ratio of the left pieces ?



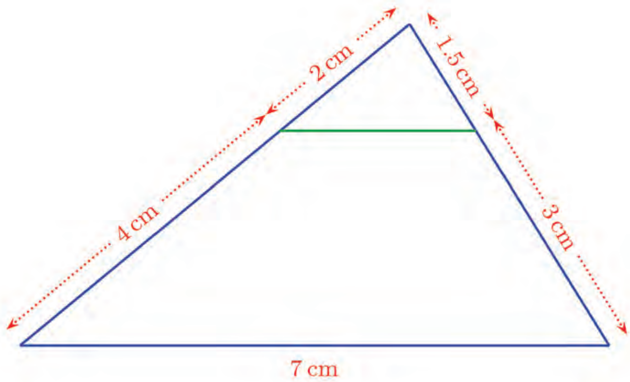
> Construct a slider a with Min = 0 and Max = 1 and Increment = 0.01. Draw a triangle *ABC*. Select **Enlarge from Point (Dilate from Point)** and click on *C* and then on *A*. In the dialogue window, type the Scale Factor as a. We get a point *C′* on the line *AC*. The length of *AC′* would be a times that of *AC*. (If a = 0.5, then *C’* would be the midpoint of *AC*. If a = 0.1, then the lengths of *AC′* and *CC′* would be in the ratio 1 : 9. Mark the lengths of these lengths and check.) Draw a line parallel to *AB* through *C′* and mark the line where it meets *BC* as *D*. Mark the lengths *BD* and *DC*. Compare the ratios *AC′* : *C′C* and *BD*: *DC*.

The longer piece on the left is twice the shorter piece. In other words, the ratio of the pieces is 1:2

So, the right side is also cut in this ratio.

Length of the shorter piece $= 4.5 \times \frac{1}{3} = 1.5$ cm

Length of the longer piece $= 4.5 \times \frac{2}{3} = 3$ cm



Did you notice another thing here?

In both the left and right sides, the shorter piece is $\frac{1}{3}$ of the whole side; and the longer piece is $\frac{2}{3}$ of the whole side.

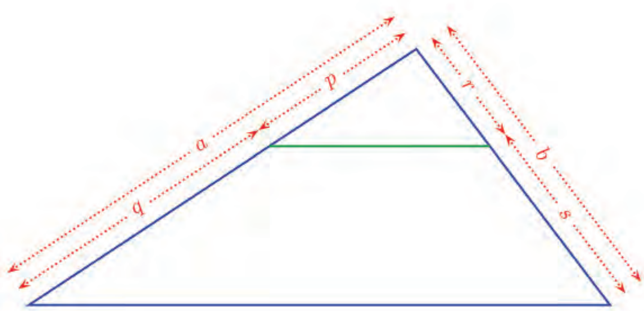
What if both sides are cut in the ratio 3: 5 ?

In both sides, the shorter piece would be $\frac{3}{8}$ of the side and the longer piece $\frac{5}{8}$ of the side.

So, the result stated earlier can also be put this way:

In any triangle, a line parallel to one side cuts the other two sides into the same parts.

See this picture:



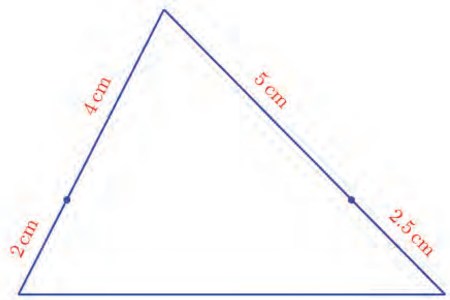
The green line within the triangle is parallel to the bottom side. Denoting the lengths of the sides and their parts by letters as in the picture, we can write the statement above as:

$$\frac{p}{a} = \frac{r}{b} \qquad \frac{q}{a} = \frac{s}{b}$$

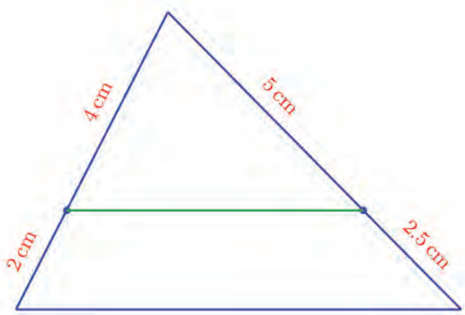
We've seen that a line parallel to one side of a triangle cuts the other two sides in the same ratio. Is the statement in reverse true?

That is, if a line divides two sides of a triangle in the same ratio, can we say that it is parallel to the third side?

For example, in the picture below, points dividing the left and right sides in the ratio 1 : 2 are marked:

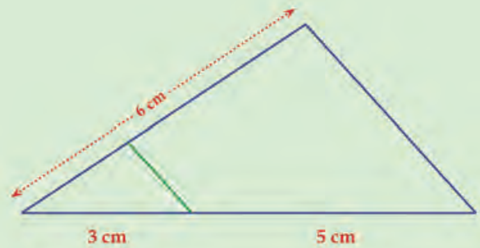


The line through the point on the left line, parallel to the bottom side, must divide the right side also in the ratio 1: 2. So, it must pass through the point marked on the right side. This means the line parallel to the bottom side, through the point on the left is the line joining these points. That is to say, the line joining these points is parallel to the bottom sides.



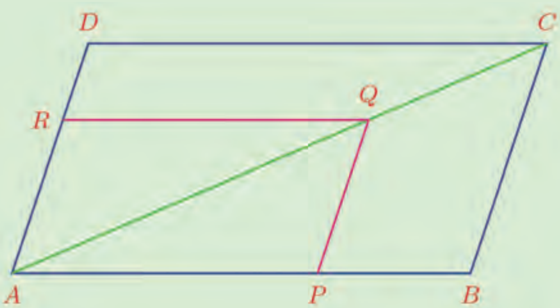
In any triangle, a line dividing two sides in the same ratio is parallel to the third side.

(1) In the picture below, the green line is parallel to the right side of the blue triangle.



Calculate the lengths of the pieces into which this line cuts the left side.

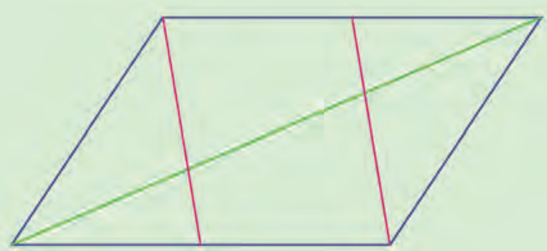
(2) In the parallelogram $ABCD$, the line through the point P on AB, parallel to BC meets AC at Q. The line through Q parallel to AB meets AD at R:



(i) Prove that $\frac{AP}{PB} = \frac{AR}{RD}$

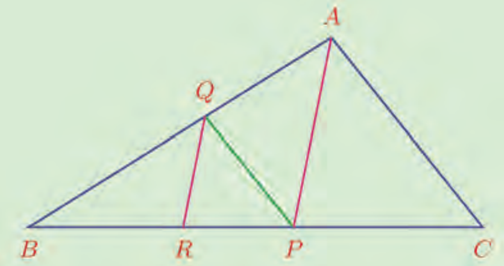
(ii) Prove that $\frac{AP}{AB} = \frac{AR}{AD}$

(3) In the picture below, two corners of a parallelogram are joined to the midpoints of two sides.



Prove that these lines cut the diagonal in the picture into three equal parts.

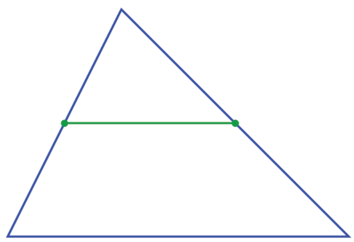
(4) In triangle ABC, the line parallel to AC through the point P on BC, meets AB at Q. The line parallel to AP through Q meets BC at R:



Prove that $\frac{BP}{PC} = \frac{BR}{RP}$

Midsection

See this picture:



The green line within the triangle is the line parallel to the bottom side, through the midpoint of the left side.

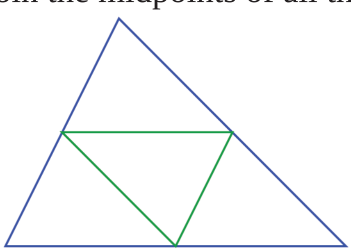
Since it cuts the left side into equal pieces, it must also cut the right side into equal pieces. In other words, it meets the right side at its midpoint.

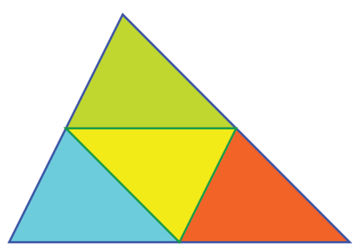
On the other hand, the line joining the midpoints of the left and right sides cuts both sides in the same ratio 1 : 1 and so is parallel to the bottom side.

This also is something which deserves special mention:

> In any triangle, the line parallel to a side, through the midpoint of another side, meets the third side also at its midpoint.
>
> On the other hand in any triangle, the line joining the midpoints of two sides is parallel to the third side

What if we join the midpoints of all three sides?





Draw a triangle in GeoGebra and mark the midpoints of the sides. Draw the triangle connecting these midpoints. Mark the lengths of the sides of the large and small triangles. What is the relation between these lengths? Move the corners of the large triangle and check.

What can we say about the four small triangles? The sides of the yellow triangle in the middle are parallel to the sides of the large triangle containing all the small triangles.

Don't all these four triangles seem to be equal? Let's check this :

First look at the blue and yellow triangles. The right side of the blue one and the left side of the yellow triangle are the same.

The angle in the blue triangle at the top of this line, and the angle in the yellow triangle at the bottom of this line are equal (why?); so are the bottom angle of the blue triangle and the top angle of the yellow triangle.

So, these two triangles are equal. In the same way, we can see that the green and red triangles are also equal to the yellow triangle. Thus, all the four triangles are equal.

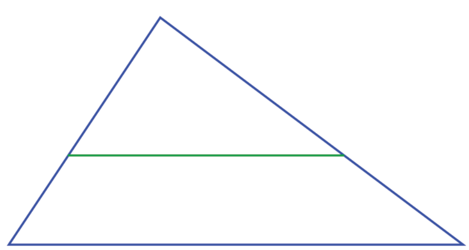
We can also see another thing from this. Since the sides of all these triangles are the same, each side is half the length of a side of the large triangle. Thus we have this result:

In any triangle, the length of the line joining the midpoints of two sides is half the length of the third side.

We've also seen that the line joining the midpoints of two sides is parallel to the third side. This, together with the above result gives this:

In any triangle, the line joining the midpoints of two sides is parallel to the third side and is half its the length.

We can look at this result in another manner. A line parallel to one side of a triangle makes a smaller triangle within the original one.



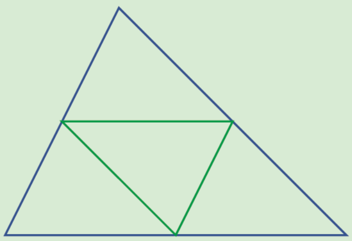
Two sides of this smaller triangle are on two sides of the large triangle; and they are the same parts of these sides.

What we've seen just now is that, if each of these sides is half of the side of the large triangle, then so is the third side.

What if the line cuts each of the two sides into a third? Would the remaining side of the smaller triangle also be a third of the remaining side of the large triangle?

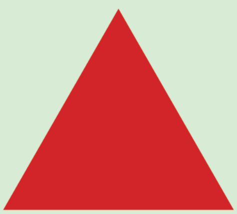
We will later see that whatever be the part of the two sides that the line cuts them, its length would be the same part of the third side.

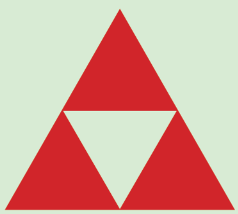
(1) In the picture, the midpoints of a triangle are joined to form a smaller triangle inside:



How many times the perimeter of the small triangle is the perimeter of the large triangle? What about their areas?

(2) See these pictures:





The first picture is that of a triangle cut out from a sheet of paper. The second one shows it with the small triangle in the middle, joining the midpoints of the sides, cut off from the large triangle.

Quarter problem

Each small triangle in the picture below is $\frac{1}{4}$ of the large triangle:

The three triangles at the bottom make $\frac{3}{4}$ of the large triangle and the yellow one at the top is the remaining $\frac{1}{4}$. What about this picture?

The blue, red and green together make $3\left(\frac{1}{4}+\frac{1}{4^2}\right)$, the yellow is $\frac{1}{16}$. And in this picture?

The blue, red and green triangles gradually fill up the whole triangle, right ? In arithmetical language this means the numbers,

$$\frac{3}{4};\ 3\left(\frac{1}{4}+\frac{1}{4^2}\right),\ 3\left(\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}\right)$$

and so on get closer and closer to 1. So, the numbers

$$\frac{1}{4},\ \frac{1}{4}+\frac{1}{4^2},\ \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}$$

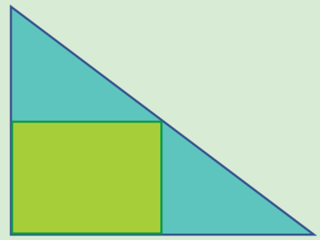
and so on get nearer and nearer to $\frac{1}{3}$.

The third picture shows such middle pieces cut off from each of the small triangles in the second picture

(i) What fraction of the area of the paper in the first picture is the area of the paper in the second picture?

(ii) What about in the third picture?

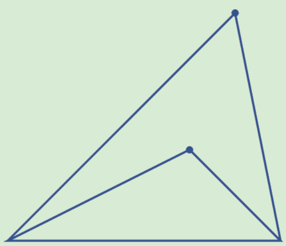
(3) A quadrilateral is drawn with the midpoints of the sides of a right triangle and its square corner as vertices.

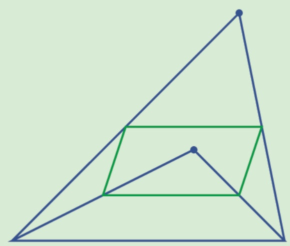


(i) Prove that this quadrilateral is a rectangle.

(ii) What fraction of the area of the triangle is the area of the rectangle?

(4) In the two pictures below, the first one shows two triangles formed by joining the ends of a line to two points above it. The second one shows the quadrilateral formed by joining the midpoints of the left and right sides of these triangles:





Draw a line AB and mark two pints C and D above it. Draw two triangles by joining these points to A and B. Mark the midpoints of the left and right sides of these triangles and draw the quadrilateral joining these points. What speciality of this quadrilateral do you notice ? Change the positions of C and D and see what happens. See at what positions of these points does the quadrilateral become a rhombus, rectangle or a square. Can we get these shapes if C and D are on different sides of AB ?

(i) Prove that this quadrilateral is a parallelogram.

(ii) Describe the positions of the points on top for this quadrilateral to be:

(a) Rhombus (b) Rectangle

(c) Square

(iii) Would we get all these, if one point is taken above the line and one below?

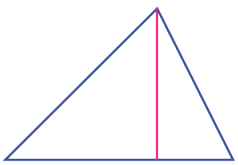
(5) (i) Prove that the quadrilateral formed by joining the midpoints of any quadrilateral is a parallelogram.

(ii) Explain what should be the original quadrilateral to get the inner quadrilateral as

(a) Rhombus (b) Rectangle (c) Square

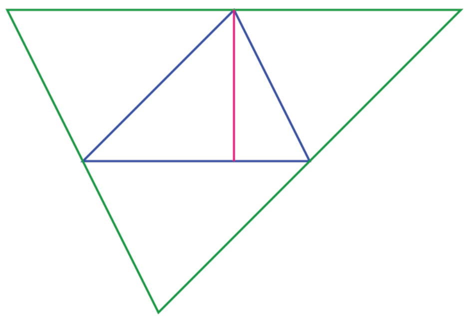
Triangle centres

See this picture:



The perpendicular from the top vertex of the triangle to the opposite side is drawn. Such perpendiculars can be drawn from any vertex to the opposite side. These perpendiculars are called the altitudes (heights) of the triangle.

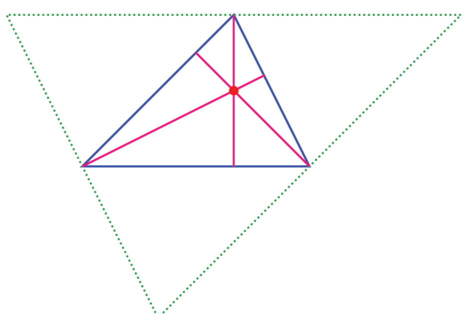
Now let's draw through each vertex, the line parallel to the opposite side:



Then the red line is not only perpendicular to the top side of the large triangle, it also bisects it.

So, if we draw the other altitudes of the smaller triangle, they would be the perpendicular bisectors of the other two sides of the larger triangle.

We have seen that all three perpendicular bisectors of the three sides of any triangle cross one another at a single point



Draw a triangle in GeoGebra and draw the perpendiculars from each vertex to the opposite side. Do they intersect at a single point ? Mark this point (orthocentre). Mark the angles of the triangle also. Now change the positions of the vertices. Is the orthocentre within the triangle in all cases ? What happens if one of the angles of the triangle is a right angle ? What if one angle is larger than a right angle ?

The perpendicular bisectors of the large triangle are the altitudes of the smaller triangle.

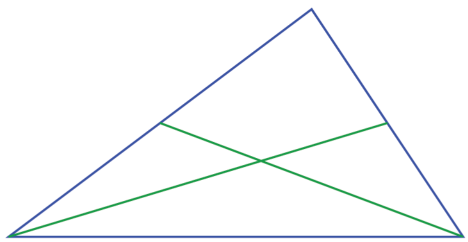
What do we see here ?

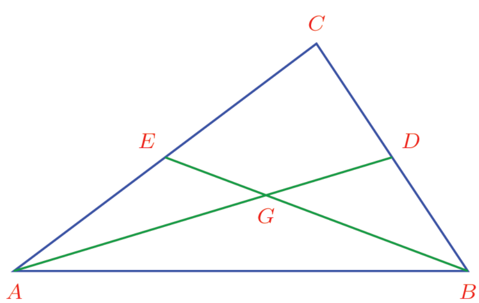
In any triangle, all the altitudes intersect at a single point.

The point where all the altitudes meet is called the *orthocentre* of the triangle.

Perpendicular bisectors of sides and altitudes are some of the special lines related to a triangle. Lines joining each vertex with the midpoint of the opposite side form another type of such lines. We have noted earlier that such lines are called medians of a triangle. Do they also intersect at a single point?

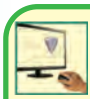
See this picture:





Two medians of a triangle are drawn. What we want to check is whether the third median also passes through the point of intersection of the two we have drawn.

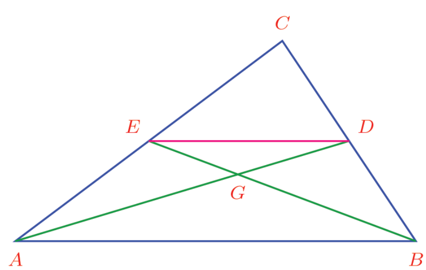
We can draw and check. But we need a proof that this is so in all triangles. To begin with, we name the points in the picture, for easy reference.



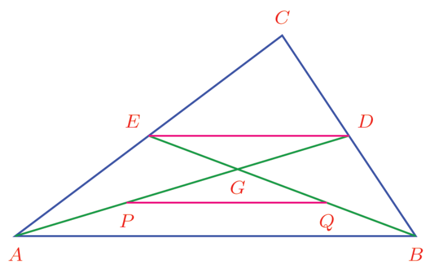
Draw a triangle in GeoGebra and mark the midpoints of the sides. Join each vertex to the midpoint of the side to get the medians. Do they all pass through a single point ? Mark this point (centroid). Mark the distance from each vertex to the centroid and also the distance from the centroid to the midpoint of the side. What is the relation between these distances? Change the positions of the vertices of triangle and check.

The line joining the midpoints of the left and right sides is parallel to the bottom side and is half its length. That is

$$ED = \frac{1}{2}AB$$



How about joining the midpoints of the small triangle GAB also?



We get

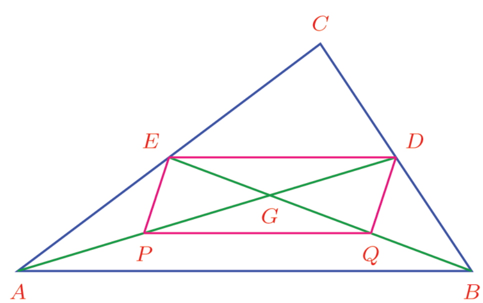
$$PQ = \frac{1}{2}AB$$

Thus we have

$$PQ = ED$$

Since the opposite sides PQ and ED of the quadrilateral $PQDE$ are parallel and equal, it is a parallelogram. So, its diagonals PD and QE bisect each other. That is,

$$PG = GD \qquad QG = GE$$



Since P is the midpoint of AG and Q is the midpoint of BG, we have

$$AP = PG \qquad BQ = QG$$

So,

$$AG = AP + PG = 2PG = 2GD$$

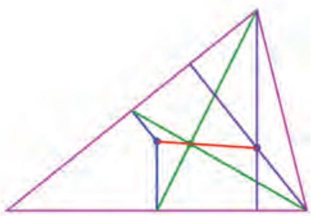
and

$$BG = BQ + QG = 2QG = 2GE$$

Draw a triangle in GeoGebra and mark the centroid (If we draw the triangle using the **Polygon** tool, then we can mark the centroid using the **Midpoint** or **Centre** tool). Draw a triangle joining the centroid and two vertices. We can draw three such triangles. Find the areas of these triangles. Is there any relation between them ? Change the positions of the vertices and check.

Euler line

The famous mathematician Euler proved in the eighteenth century that in any triangle, the circumcentre, centroid and orthocentre lie along the same straight line. This line is now known as the Euler line.



In the picture above, the blue lines are the perpendicular bisectors of two sides of the triangle. Their point of intersection is the circumcentre. The green lines are medians and their point of intersection is the centroid. The violet lines are altitudes and their point of intersection is the orthocentre.

The red line passing through these three points is the Euler line.

Euler also proves that in any triangle, the centroid is between the circumcentre and orthocentre and the centroid divides the line joining them in the ratio 1: 2.

Draw a triangle in GeoGebra and find its circumcentre, centroid and orthocentre. Draw the line joining any two of them. Does it pass through the third point also ? Mark the lengths of the sides of the triangle. Change the positions of the vertices of the triangle. What happens to the Euler line when two sides of the triangle are equal? What happens to the three points when all three sides are equal ?

This shows that the point G divides the lines AD and BE in the ratio 2 : 1.

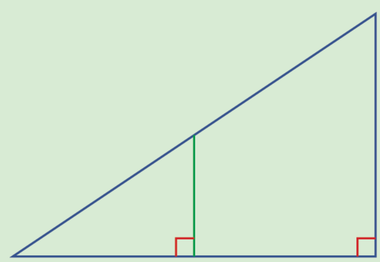
Now suppose instead of joining the vertices A and B to the midpoints of their opposite sides, we join the vertices B and C to the midpoints of their opposite sides. Their points of intersection would divide BE in the ratio 2 : 1. In other words, their point of intersection is also G.

So we get this result:

In any triangle, all three medians intersect at a single point. The distance of this point from each vertex is twice the distance from the midpoint of the opposite side. In other words, this point divides each median in the ratio 2 : 1.

This point of intersection of the medians is called the centroid of the triangle.

(1) Draw a right triangle and draw the perpendicular from the midpoint of the hypotenuse to the base:



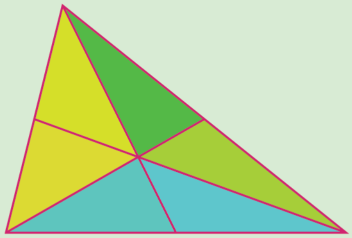
(i) Prove that this perpendicular is half the vertical side of the large triangle.

(ii) Prove that the distances from the midpoint of the hypotenuse to the three vertices of the large triangle are equal.

(iii) Prove that the circumcentre of a right triangle is the midpoint of its hypotenuse.

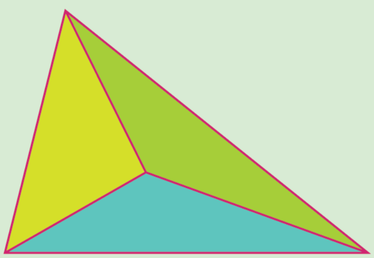
(2) Prove that in any equilateral triangle, the circumcentre, orthocentre and the centroid coincide.

(3) In the picture below, the medians of the triangle divide it into six small triangles:



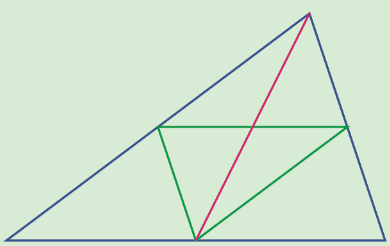
Prove that all six triangles have the same area.

(4) In the picture below, the triangle is divided into three small triangles by joining the centroid to the three vertices:



Prove that all three triangles have the same area.

(5) In the picture below, the midpoints of the sides of the blue triangle are joined to make the smaller green triangle. The red line is a median of the large triangle.



Balancing act

In physics, we often have to assume that the entire mass of a body is concentrated at a single point. It is called the centre of mass. It is physically experienced as the point at which the body can be balanced.

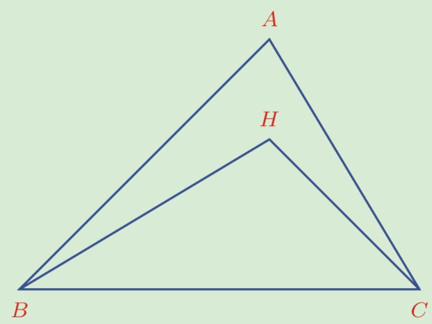
The centre of mass of a triangle cut out from a thick sheet of paper or a thin sheet of metal is its centroid.

Draw a triangle in GeoGebra and mark the midpoints of the sides. Draw the altitudes from each vertex and mark the points where they meet the opposite sides. Find the orthocentre of the triangle. Mark the midpoints of the lines joining the orthocentre and each vertex. Draw the circle through these three points (The **Circle through 3 Points** tool can be used for this). Aren't the other six points we marked, the midpoints of the sides and the feet of the altitudes, also on this circle ? This circle which passes through nine points of the triangle is called its Nine-point circle.

(i) Prove that this median bisects the top side of the small triangle.

(ii) Prove that the centroid of the large and small triangles are the same.

(6) In the picture below, H is the orthocentre of triangle ABC.



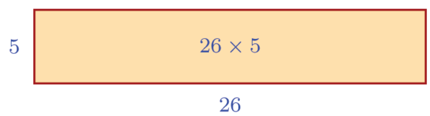
Prove that A is the orthocentre of triangle HBC

4

Multiplication Identities

Product of sums

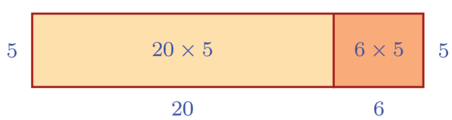
What is the area of a rectangle of sides 26 centimetres and 5 centimetres?

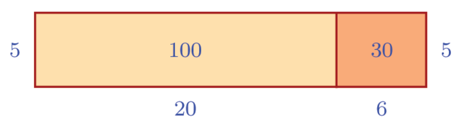


Instead of multiplying directly, we can split the product like this:

$$26 \times 5 = (20 + 6) \times 5$$

We can also split the rectangle accordingly:





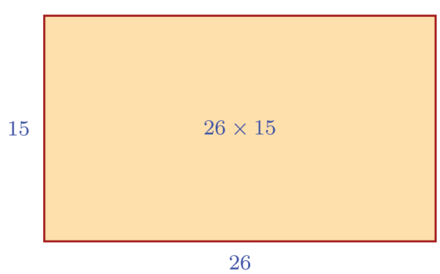
So area is

$$100 + 30 = 130 \text{ sq.cm}$$

How was the multiplication done here?

$$\begin{aligned} 26 \times 5 &= (20 + 6) \times 5 \\ &= (20 \times 5) + (6 \times 5) \\ &= 100 + 30 \\ &= 130 \end{aligned}$$

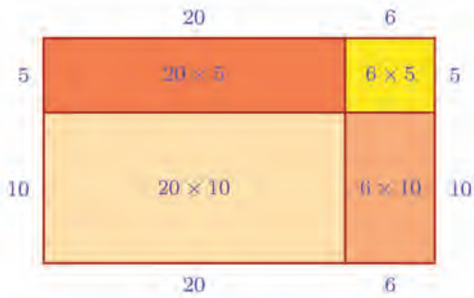
Now let's look at a rectangle of sides 26 centimetres and 15 centimetres:

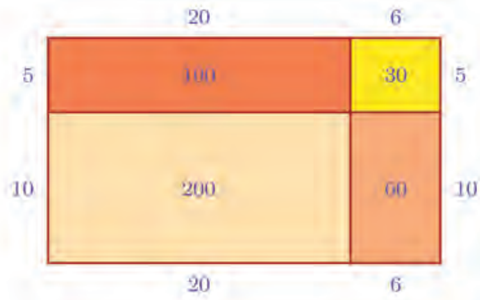


Here also, if we split the product as,

$$26 \times 15 = (20 + 6) \times (10 + 5)$$

and split the rectangle also according to this, the computations would be a lot easier:





Now we can compute the area as

$$200 + 100 + 60 + 30 = 390 \text{ sq.cm.}$$

Let's write out how we got these numbers to be added:

$$\begin{aligned} 26 \times 15 &= (20 + 6) \times (10 + 5) \\ &= (20 \times 10) + (20 \times 5) + (6 \times 10) + (6 \times 5) \\ &= 200 + 100 + 60 + 30 \\ &= 390 \end{aligned}$$

Isn't this how we find the product when we write the numbers one below the other and multiply?

$$\begin{array}{l} 26\ \times \\ \underline{15} \\ 130 \leftarrow 26 \times 5 = (6 + 20) \times 5 = 30 + 100 \\ \underline{260} \leftarrow 26 \times 10 = (6 + 20) \times 10 = 60 + 200 \\ 390 \end{array}$$

The operations done here can be explained like this.

To multiply the sum 20 + 6 by the sum 10 + 5, we multiplied each of the numbers 20 and 6 in the first sum by each of the numbers 10 and 5 in the second sum and added together, all the four products thus got.

This works even if the numbers are not natural numbers.

For example,

$$6\tfrac{1}{2} \times 8\tfrac{1}{3} = \left(6+\tfrac{1}{2}\right) \times \left(8+\tfrac{1}{3}\right)$$
$$= (6 \times 8) + \left(6 \times \tfrac{1}{3}\right) + \left(\tfrac{1}{2} \times 8\right) + \left(\tfrac{1}{2} \times \tfrac{1}{3}\right)$$
$$= 48 + 2 + 4 + \tfrac{1}{6}$$
$$= 54\tfrac{1}{6}$$

In general,

To multiply a sum by a sum, we must multiply each number in the first sum by each number in the second sum and add all these products together.

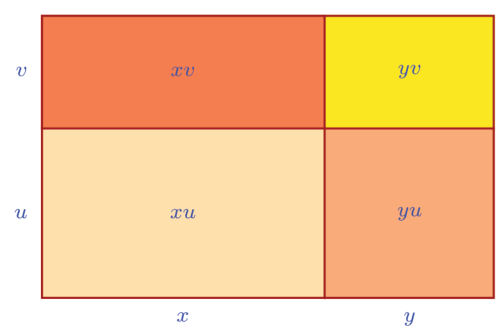
How about writing this in algebra?

Let's take the first sum as $x + y$ and the second sum as $u + v$. To find their product, we must add xu, xv, yu, yv. Thus

For any positive numbers x, y, u, v

$$(x + y)(u + v) = xu + xv + yu + yv$$

This can be seen as areas like this:



This can be used to simplify some computations.

For example, look at 31×51

We can do $30 \times 50 = 1500$ in head.

By the result seen just now,

$$31 \times 51 = (30 + 1) \times (50 + 1) = 1500 + 30 + 50 + 1 = 1581$$

and this can also be done mentally.

What is the algebraic equation used here?

$$(x + 1)(y + 1) = xy + x + y + 1$$

In other words, knowing the product of two natural numbers, to get the product of the numbers next to each, we need only add the product of the first numbers, their sum and then one more.

For example,

$$21 \times 71 = (20 \times 70) + 20 + 70 + 1 = 1400 + 91 = 1491$$

$$81 \times 91 = (80 \times 90) + 80 + 90 + 1 = 7200 + 171 = 7371$$

$$201 \times 401 = (200 \times 400) + 200 + 400 + 1 = 80000 + 601 = 80601$$

Just as this gives an easy method to compute the product of numbers increased by one, is there an easy way to compute the product of numbers increased by two? Think about it:

We can use the general result to find the product of certain fractions also:

$$\left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = xy + \left(x \times \frac{1}{2}\right) + \left(\frac{1}{2} \times y\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) = xy + \frac{1}{2}(x + y) + \frac{1}{4}$$

For example,

$$6\frac{1}{2} \times 8\frac{1}{2} = 48 + \left(\frac{1}{2} \times 14\right) + \frac{1}{4} = 48 + 7\frac{1}{4} = 55\frac{1}{4}$$

$$10\frac{1}{2} \times 5\frac{1}{2} = 50 + \left(\frac{1}{2} \times 15\right) + \frac{1}{4} = 50 + 7\frac{1}{2} + \frac{1}{4} = 57\frac{3}{4}$$

(1) Mentally find the products below:

(i) 71×91 (ii) 42×62 (iii) $10\frac{1}{2} \times 6\frac{1}{2}$ (iv) 9.5×3.5 (v) $10\frac{1}{4} \times 6\frac{1}{4}$

(2) The product of two numbers is 1400 and their sum is 81. What is the product of the numbers next to each?

(3) The product of two odd numbers is 621 and their sum is 50. What is the product of the odd numbers next to each?

Special operations

What we write as algebraic identities are general principles on numbers and their operations. Using them, we can prove some other general results also.

For example,

We can see that the product of two odd numbers is odd by inspecting some pairs of odd numbers. How do we actually prove this for any two odd numbers?

We've seen in Class 7 that the general form of an odd number is

$$2n + 1, n = 0, 1, 2, 3, ...$$

So we can take two odd numbers in general as $2m + 1$ and $2n + 1$. Their product is

$$(2m + 1)(2n + 1) = 4mn + 2m + 2n + 1$$

Is this also an odd number?

Let's rewrite this in a slightly different form:

$$(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$$

Now isn't it clear that the product is also an odd number?

Next we note that odd numbers are those numbers which leave remainder 1 on division by 2.

So, what is the general form of a number which leaves remainder 1 on division by 3?

Such numbers are got by adding 1 to a multiple of 3, right?

This thought leads to their general form:

$$3n + 1, n = 0, 1, 2, 3, ...$$

What about the product of two such numbers?

$$(3m + 1)(3n + 1) = 9mn + 3m + 3n + 1$$

Does the product also leave remainder 1 on division by 3? To see it, we need only write the product in a slightly different form:

$$(3m + 1)(3n + 1) = 3(3mn + m + n) + 1$$

This also is 1 added to a multiple of 3, isn't it?

In other words, a number which leaves remainder 1 on division by 3.

Let's look at another kind of problem:

If we take the first four natural numbers, 1, 2, 3, 4

Product of the two numbers at the ends = $1 \times 4 = 4$

Product of the two numbers in the middle = $2 \times 3 = 6$

Let's try this with next four, 2, 3, 4, 5 ?

Product of the two numbers at the ends = $2 \times 5 = 10$

Product of the two numbers in the middle = $3 \times 4 = 12$

Try some other four consecutive numbers. Is the difference of the products 2 every time? How do we prove that this is true for any four consecutive natural numbers?

Four consecutive natural numbers in general can be taken as x, $x + 1$, $x + 2$, $x + 3$

Product of the two numbers at the ends = $x(x + 3) = x^2 + 3x$

Product of the two numbers in the middle = $(x + 1)(x+2) = x^2 + 3x + 2$

This shows the difference is 2 for all such products.

In these equations, we can take any number as x. For natural numbers, we can state this property in words like this:

For any four consecutive natural numbers, the difference between the products of the two numbers at the ends and the two numbers in the middle is two

(1) Calculate each of the following for several numbers and guess a generalization; then prove it using algebra.

(i) The remainder got on division by 3, the product of two numbers one of which leaves remainder 1 on division by 3 and the other, remainder 2.

(ii) The remainder got on division by 4, the product of two numbers one of which leaves remainder 1 on division by 4 and the other, remainder 2.

(iii) The difference of the products of the two numbers at the ends and the two in the middle, of six consecutive natural numbers.

(2) Given below is a method to find the product 36 × 28

$3 \times 2 = 6$	6×100	600
$(3 \times 8) + (6 \times 2) = 36$	36×10	360
6×8		48
36×28		1008

(i) Check these for some other products of two digit numbers.

(ii) Explain why this works using algebra.

(Recall the general algebraic form $10m + n$ of a two digit number, seen in Class 7)

Number curiosities

We can use algebra to explain some curious properties of numbers also. For example, look at these products:

$$12 \times 63 = 756$$

$$21 \times 36 = 756$$

Usually the product of a pair of two digit numbers and the product of these with their digits reversed are different (try it!) Are there other pairs giving the same product ?

Look at these products:

$$32 \times 46 = 1472$$

$$23 \times 64 = 1472$$

Are there other pairs?

Let's use algebra

We know that any two digit number is of the form $10m + n$

First digit is m and the second digit n

If we reverse the digits?

Then the first digit is n and the second, m; and the number itself is $10n + m$.

We want a pair of two digit numbers. we take them as $10m + n$ and $10p + q$.These with the digits reversed are $10n + m$ and $10q + p$.

We want to choose m, n, p, q in such a way that the two pairs have the same product.

The product of the first two is

$$(10m + n)(10p + q) = 100mp + 10mq + 10np + nq$$

The products with digits reversed is

$$(10n + m)(10q + p) = 100nq + 10np + 10mq + mp$$

Each product is a sum of four numbers. Comparing them, we see that both sums have $10mq + 10np$. So for the sums to be equal, the sum of the remaining two must be equal. That is,

$$100mp + nq = 100nq + mp$$

What does this mean?

Can the sum of hundred times a number and another be equal to the sum of hundred times the second and the first ?

So, the products mp and nq must be equal.

$$mp = nq$$

We can also see this using algebra. If the numbers $100mp + nq$ and $100nq + mp$ are to be equal, their difference must be zero. That is,

$$(100mp + nq) - (100nq + mp) = 0$$

This gives

$$99mp - 99nq = 0$$

This we can write

$$99(mp - nq) = 0$$

If 99 times a number is zero, the number itself must be zero, right ? So

$$mp - nq = 0$$

and this means

$$mp = nq$$

So in any pair of two digit numbers we are seeking, the products of the first digits must be equal to the product of the second digits.

In the first example above we had 12 and 63

Product of the first digits = $1 \times 6 = 6$

Product of the second digits = $2 \times 3 = 6$

What about the second example ? Numbers are 32 and 46

Product of the first digits = $3 \times 4 = 12$

Product of the second digits = $2 \times 6 = 12$

Similarly, since

$$2 \times 9 = 18$$
$$3 \times 6 = 18$$

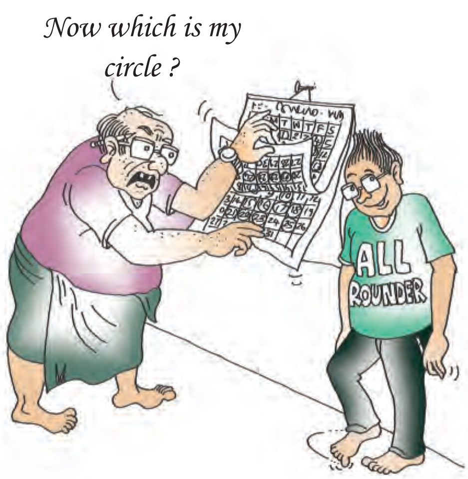
the products 23 × 96 and 32 × 69 must be equal (check).

Can you find other pairs?

Remember some interesting things we found in Class 7 about sums in a calendar ? Now let’s see something about products of these numbers

Take any month in a calendar and mark four dates which form a square:

SUN	MON	TUE	WED	THU	FRI	SAT
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	



Multiply the diagonal pairs:

$$14 \times 6 = 84$$

$$13 \times 7 = 91$$

And find their difference:

$$91 - 84 = 7$$

Now take another set of four numbers like these:

$$22 \times 30 = 660$$

$$23 \times 29 = 667$$

$$667 - 660 = 7$$

Why is the difference 7 every time?

Let’s use algebra to find out.

If we denote the first number of the selected square as x, then the four numbers are like this:

x	$x+1$
$x+7$	$x+8$

The diagonal products are $x(x + 8)$ and $(x + 1)(x + 7)$

The first product is

$$x(x + 8) = x^2 + 8x$$

And the second product

$$(x + 1)(x + 7) = x^2 + 7x + x + 7 = x^2 + 8x + 7$$

Look at the two products. The difference is 7, isn't it ?

We can take x as any number, so it is always true for any other part of a calender.

Let's look at another one

Make a multiplication table like this:

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

As we did in the calendar, mark four numbers in a square at different positions:

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

Instead of multiplying as before, add the diagonal pairs in each square:

$12 + 20 = 32$ $\qquad$ $35 + 48 = 83$

$16 + 15 = 31$ $\qquad$ $40 + 42 = 82$

Will the difference of the sums be the same, whatever be the position of the square ?

To check this using algebra, we need to know the general form of the numbers so chosen from the table.

The first row of the table is just the natural numbers from 1 to 9, the second row is these numbers multiplied by 2, . . .

In general, for any natural number n from 1 to 9, the numbers in the n^{th} row are

$$n \quad 2n \quad 3n \quad 4n \quad 5n \quad 6n \quad 7n \quad 8n \quad 9n$$

What about the next line?

To get it, by what number should we multiply the first row?

Let's look at these two rows together:

n	$2n$	$3n$	$4n$	$5n$	$6n$	$7n$	$8n$	$9n$
$n + 1$	$2(n + 1)$	$3(n + 1)$	$4(n + 1)$	$5(n + 1)$	$6(n + 1)$	$7(n + 1)$	$8(n + 1)$	$9(n + 1)$

Now to take four numbers in a square, we can start with any number in the first row above. For example, we can choose like these:

$2n$	$3n$
$2(n+1)$	$3(n+1)$

$5n$	$6n$
$5(n+1)$	$6(n+1)$

In general, if we start with the m^{th} multiple of n, what would be the numbers in the square?

Let's write them one by one:

mn	$(m+1)n$

mn	$(m+1)n$
$m(n+1)$	

mn	$(m+1)n$
$m(n+1)$	$(m+1)(n+1)$

Expanding all products, we get

mn	$mn+n$
$mn+m$	$mn+m+n+1$

In this, one diagonal sum is

$$mn + (mn + m + n + 1) = 2mn + m + n + 1$$

And the other sum is

$$(mn + n) + (mn + m) = 2mn + n + m$$

Now don't you see that the difference of the sums is 1, wherever the square be, and also why it is so?

(1) Write numbers as shown below:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

(i) As we did in the calendar, mark a square of four numbers and find the difference of the diagonal products. Do we get the same difference for any such square?

(ii) Explain why this is so, using algebra.

(2) In the multiplication table we've made, draw a square of nine numbers, instead of four, and mark the numbers at the four corners:

6	8	10
9	12	15
12	16	20

(i) What is the difference of the diagonal sums ?

(ii) Explain using algebra why we get the difference as 4 in all such squares.

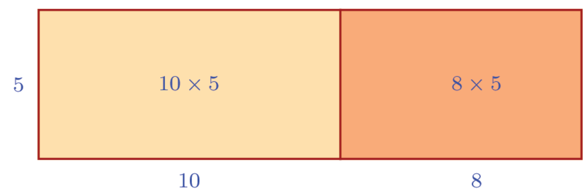
(iii) What about a square of sixteen numbers?

Multiplication with a difference

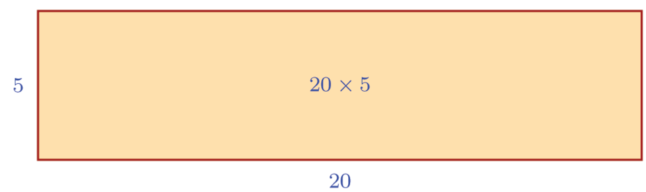
How do we calculate 18×5 ?

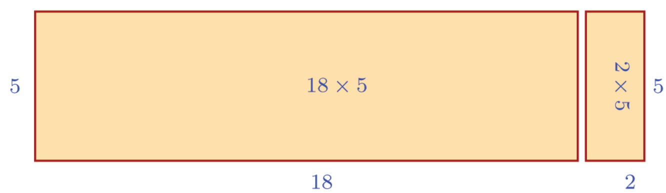
We can do as before:

$$18 \times 5 = (10 + 8) \times 5 = (10 \times 5) + (8 \times 5) = 50 + 40 = 90$$



There's another way:



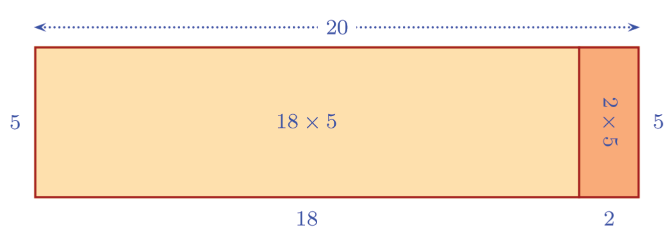


That is,

$$\begin{aligned} 18 \times 5 &= (20 - 2) \times 5 \\ &= (20 \times 5) - (2 \times 5) \\ &= 100 - 10 \\ &= 90 \end{aligned}$$

We can also look at it like this: 18×5 and 2×5 added together gives 20×5:

$$(18 \times 5) + (2 \times 5) = (18 + 2) \times 5 = 20 \times 5$$



So,

$$18 \times 5 = (20 \times 5) - (2 \times 5)$$

Like this, we can compute

$$38 \times 5 = (40 - 2) \times 5 = (40 \times 5) - (2 \times 5) = 200 - 10 = 190$$

and

$$45 \times 8 = 45 \times (10 - 2) = (45 \times 10) - (45 \times 2) = 450 - 90 = 360$$

What did we do in these two calculations? Just as we split the product of a sum by a number, we can also split the product of a difference by a number.

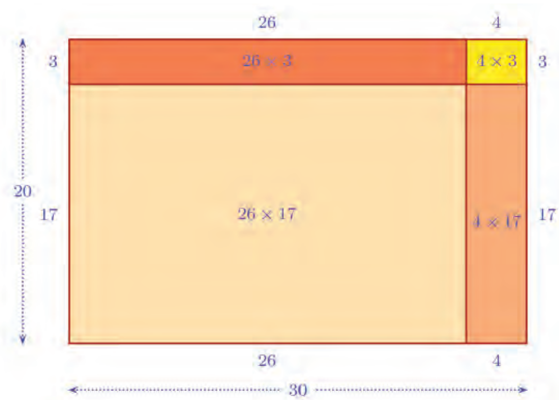
Now let's see whether we can split the product of two differences, as we did for the product of sums.

For example we can write,

$$26 \times 17 = (30 - 4) \times (20 - 3)$$

How do we split this as four products as in the case of sums?

The easy way to do this is to think how we can enlarge a rectangle of sides 26 and 17 to one with sides 30 and 20:



From this picture, we can see that

$$(26 \times 17) + (26 \times 3) + (4 \times 17) + (4 \times 3) = 30 \times 20$$

And from this equation, we can see that

$$26 \times 17 = 30 \times 20 - (26 \times 3) - (4 \times 17) - (4 \times 3)$$

Now let’s calculate each product on the right side of the equation:

$$30 \times 20 = 600$$

$$26 \times 3 = (30 - 4) \times 3 = 90 - 12 = 78$$

$$4 \times 17 = 4 \times (20 - 3) = 80 - 12 = 68$$

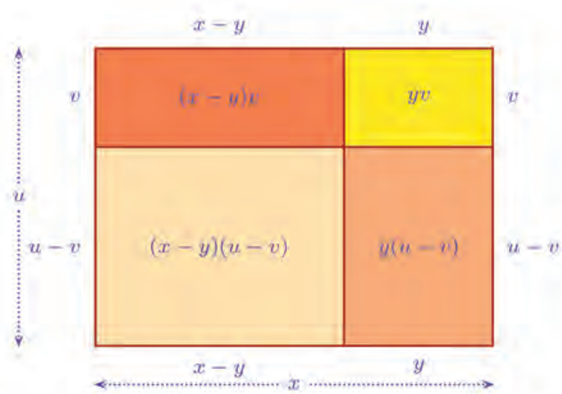
$$4 \times 3 = 12$$

Thus

$$\begin{aligned}(26 \times 17) &= 600 - 78 - 68 - 12 \\ &= 600 - (78 + 68 + 12) \\ &= 600 - 158 \\ &= 442\end{aligned}$$

The next thing we think about is whether we can do this without so much computation. For that we use algebra without actual numbers.

The problem is to split $(x - y)(u - v)$. As in the problem above, we start with a rectangle of sides $x - y$ and $u - v$ and enlarge it to one with sides x and u:



From the picture we see that

$$(x - y)\,(u - v) + y(u - v) + (x - y)v + yv = xu$$

Expanding the middle two products in the sum on the left of the equation, we get

$$(x - y)(u - v) + (yu - yv) + (xv - yv) + yv = xu$$

This simplifies to

$$(x-y)(u-v)+yu-yv+xv=xu$$

This says that to increase $(x-y)(u-v)$ to xu, we must add xv and yu and subtract yv (Can you see this in the picture?)

So to change xu back to $(x-y)(u-v)$, the added xv and yu must be subtracted and the subtracted yv must be added back. That is,

$$(x-y)(u-v)=xu-xv-yu+yv$$

Thus we get another general result about multiplication:

For any positive numbers x, y, u, v with $x > y$ and $u > v$

$$(x-y)(u-v)=xu-xv-yu+yv$$

Using this, our first problem can be done much more easily:

$$\begin{aligned}26 \times 17 &= (30-4) \times (20-3)\\ &= (30 \times 20)-(30 \times 3)-(4 \times 20)+(4 \times 3)\\ &= 600-90-80+12\\ &= 612-170\end{aligned}$$

To subtract 170 from 612, the easy way is to subtract 200 and add 30. Thus

$$26 \times 17 = 612-200+30 = 442$$

Now calulate these products like this:

(i) 38×49 (ii) 47×99 (iii) 29×46 (iv) $9\frac{1}{2} \times 19\frac{1}{2}$

Let's look at another problem :

Remember how from the product of two natural numbers,we calculated the product of the next numbers? We did it using the algebraic form of the product of two sums. Like this, using the equation of the product of differences, we can find the product of the numbers just before.

For example,

$$29 \times 49 = (30 - 1) \times (50 - 1)$$
$$= (30 \times 50) - (30 \times 1) - (50 \times 1) + (1 \times 1)$$
$$= 1500 - 30 - 50 + 1$$
$$= 1500 - 80 + 1$$
$$= 1421$$

This computation can be given a general algebraic form:

$$(x - 1)(y - 1) = xy - x - y + 1$$

and we can rewrite it like this:

$$(x - 1)(y - 1) = xy - (x + y) + 1$$

This means that if we know the product of two natural numbers, then to calculate the product of the numbers just before each, we need only subtract the sum from the known product and add one.

On the other hand, we can compute the product of two numbers, if we know the product of the numbers just after and the product of the numbers just before each.

For example, suppose that the product of the numbers just after two numbers is 525 and the product of the numbers just before is 427.

Taking the numbers as x and y, these give

$$(x + 1)(y + 1) = 525$$
$$(x - 1)(y - 1) = 437$$

Expanding the products on the left of the equation, we get

$$xy + x + y + 1 = 525$$
$$xy - (x + y) + 1 = 437$$

Adding these two equations, we get

$$2xy + 2 = 962$$

This gives

$$xy = \frac{1}{2}(962 - 2) = 480$$

which is the product of the numbers.

Now instead of adding the equations, we subtract the second from the first. This gives

$$2(x + y) = 88$$

so that

$$x + y = 44$$

Thus we get the sum of the numbers also.

From the product and the sum, we can also get the difference. Recall the identity

$$(x - y)^2 = (x + y)^2 - 4xy$$

seen in Class 8

So in our problem,

$$\begin{aligned}(x - y)^2 &= 44^2 - (4 \times 480)\\ &= (4^2 \times 11^2) - (4^2 \times 120)\\ &= (4^2 \times 121) - (4^2 \times 120)\\ &= 4^2\end{aligned}$$

which gives

$$x - y = 4$$

From the sum and difference we can calculate the numbers themselves:

$$x = \frac{1}{2}(44 + 4) = 24$$

$$y = \frac{1}{2}(44 - 4) = 20$$

(1) The perimeter of a rectangle is 40 centimetres and its area is 70 square centimetres. Find the area of the rectangle with each side 3 centimetres shorter.

(2) If the sides of a rectangle are decreased by one metre, its area would be 741square metres; if increased by one metre, it would be 861 square metres.

(i) What is the area of the rectangle ?

(ii) What is its perimeter ?

(iii) What are the lengths of its sides ?

(3) When each of two numbers are increased by one, the product becomes 1271 and when each is decreased by one, the product becomes 1131.

(i) What is the product of the numbers ?

(ii) What is their sum ?

(iii) What are the numbers ?

(4) The product of two odd numbers just after each of two odd numbers is 285 and the product of the odd numbers just before each is 165. What are the numbers?

We have seen identities for the product of two sums and the product of two differences. What about the product of a sum and a difference?

In other words, how do we split $(x + y)(u - v)$?

Let's continue with algebra. First we split the product like this:

$$(x + y)(u - v) = x(u - v) + y(u - v)$$

Next we expand each product on the right side of the equation:

$$x(u - v) = xu - xv$$

$$y(u - v) = yu - yv$$

$$\text{Thus} \quad (x + y)(u - v) = xu - xv + yu - yv$$

For any positive numbers x, y, u, v with $u > v$

$$(x + y)(u - v) = xu - xv + yu - yv$$

As an example let's do 14×59,

$$\begin{aligned} 14 \times 59 &= (10 + 4) \times (60 - 1) \\ &= 600 - 10 + 240 - 4 \\ &= 590 + 236 \\ &= 826 \end{aligned}$$

Now try these:

(i) 52×19 (ii) 101×48 (iii) 97×102 (iv) $9\frac{3}{4} \times 20\frac{1}{2}$

Remember how we used the sum and product of two numbers to find the products of these numbers increased by one and the product of these decreased by one?

What about the product of one number increased by one and the other decreased by one?

Let's check using 8 and 5.

$$\begin{aligned} 8 \times 5 &= 40 \\ (8 + 1) \times (5 - 1) = 9 \times 4 &= 36 \\ (8 - 1) \times (5 + 1) = 7 \times 6 &= 42 \end{aligned}$$

Experiment with other pairs of numbers. Can you find anything in general?

If the larger of the numbers is increased by one and the smaller decreased by one, would the product increase or decrease?

And the other way round?

Let's use algebra. Denote the larger number by x and the smaller number by y.

If the larger number is increased by one and the smaller number decreased by 1, the product is

$$(x + 1)(y - 1) = xy - x + y - 1$$

This means from the original product, the larger number x is subtracted and the smaller number y is added. Then itself the product becomes less; and one also is subtracted.

Thus the product is decreased.

What if it's the other way round ?

$$(x - 1)(y + 1) = xy + x - y - 1$$

The larger number x is added and the smaller number y is subtracted, which increases the product; but a one is subtracted.

So here there are two cases:

- If the numbers are consecutive, there would be no change in the product.
- If the numbers are different and not consecutive, the product would increase.

Think why these are so.

(1) The product of two numbers is 713 and their difference is 8.

(i) What is the product of the larger number increased by one and the smaller number decreased by one ?

(ii) What is the product of the larger number decreased by one and the smaller number increased by one ?

(2) The product of two numbers is 5 more than the product of the larger of the numbers increased by one and the smaller decreased by one. How much is the increase in the product if the larger is decreased by one and the smaller increased by one?

(3) The product of the larger of two numbers increased by one and the smaller decreased by one is 540. The product of the larger decreased by one and the smaller increased by one is 560.

(i) What is the product of the numbers themselves ?

(ii) What is their difference ?

(iii) What are the numbers ?

(4) If the length of a rectangle is increased by 3 metres and the breadth decreased by 2 metres, its area would decrease by 10 square metres. If the length is decreased by 2 metres and the breadth increased by 3 metres, the area would increase by 30 square metres. Calculate the length and breadth of the rectangle.

Irrational Multiplication

5

We have seen that the length of the diagonal of a square cannot be expressed as a fractional multiple of the length of its side. In other words, if we take the side of a square as the unit of length, then its diagonal cannot be expressed as a fraction.

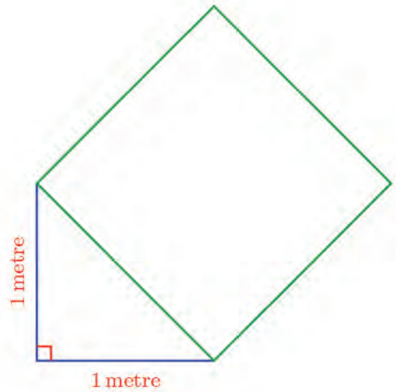
We have also seen that this length is denoted by the symbol $\sqrt{2}$. We then saw some other lengths like these and new numbers to denote them. (We will see another such number in the lesson, **Circular Measures**).

Such numbers which are used to denote lenghts which cannot be expressed as fractions of a unit are called *irrational numbers*.

We have also seen how joining lengths denoted by irrational numbers leads to the definition of addition of such numbers. We now see how computation of areas lead to the operation of multiplication of such numbers.

Multiplication

We've seen this picture so many times:



What is the perimeter of the square in this?

We know that the length of each of its sides is $\sqrt{2}$ metres. So, its perimeter is four times this length.

That is, $\sqrt{2}+\sqrt{2}+\sqrt{2}+\sqrt{2}$

As in the case of other numbers, we can write 4 times $\sqrt{2}$ as $4\times\sqrt{2}$ or $\sqrt{2}\times 4$. Usually this is written without the multiplication symbol, as $4\sqrt{2}$. Thus

$$4\sqrt{2} = \sqrt{2}+\sqrt{2}+\sqrt{2}+\sqrt{2}$$

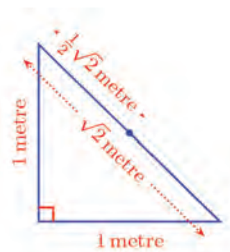
To find fractions approximating this number, we multiply by 4, fractions approximating $\sqrt{2}$:

$$4 \times 1.4, \quad 4 \times 1.41, \quad 4 \times 1.414, \ldots$$

So the perimeter of our square up to a millimetre is

$$4 \times 1.414 = 5.656 \text{ metres}$$

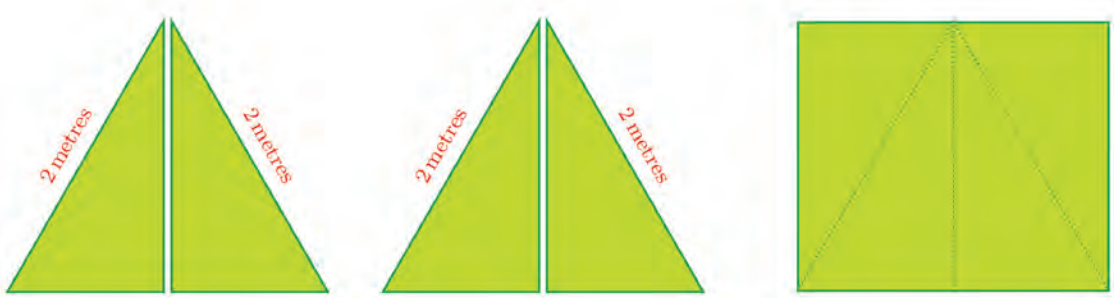
In the same way half of $\sqrt{2}$ is written as $\frac{1}{2}\sqrt{2}$.



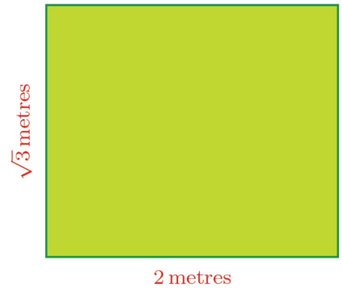
As before, to get fractions approximating $\frac{1}{2}\sqrt{2}$, we take half of fractions approximating $\sqrt{2}$. That is

$$\frac{1}{2}\sqrt{2} = 0.7071\ldots$$

Now see these pictures:



Two equilateral triangles of the same size are cut into right triangles and the pieces are rearranged to form a rectangle.



What is the area of this rectangle?

We know the length of its sides.

We've seen that if the lengths of the sides of a rectangle are fractions, then its area is the product of those numbers.

Is the area in this case also the product of its sides ? That is, $2\sqrt{3}$ square metres.

To check this, let's draw rectangle within this, all with one side 2 metres and the other sides of lengths approximating $\sqrt{3}$ metres:

As we continue with inner rectangles of heights 1.73 metres, 1.732 metres and so on, we find their areas as twice these lengths.

Geometrically these rectangles gradually fill up the outer rectangle. So, their areas get nearer and nearer to the area of the outer rectangle.

In other words, the numbers

$$2 \times 1.7, \qquad 2 \times 1.73, \qquad 2 \times 1.732, \ldots$$

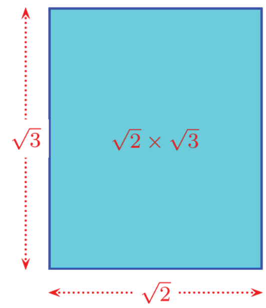
get closer and closer to the area of the outer rectangle.

According to our definition, these numbers get closer and closer to $2\sqrt{3}$.

Thus we see that the area of a rectangle of sides 2 and $\sqrt{3}$ is $2\sqrt{3}$.

Now what about a rectangle of sides $\sqrt{2}$ and $\sqrt{3}$?

We denote this area by the symbol $\sqrt{2} \times \sqrt{3}$.



To describe this as a number, we multiply the fractions approximating $\sqrt{2}$ and $\sqrt{3}$ in order and take up to the required number of decimal places:

$$\begin{array}{cccccc} 1.4 & 1.41 & 1.414 & \dots & \rightarrow & \sqrt{2} \\ 1.7 & 1.73 & 1.732 & \dots & \rightarrow & \sqrt{3} \\ \hline 2.4 & 2.44 & 2.449 & \dots & \rightarrow & \sqrt{2} \times \sqrt{3} \end{array}$$

That is,

$$\sqrt{2} \times \sqrt{3} = 2.449\dots$$

A question naturally arises here.

We know that the product of square roots, which are natural numbers or fractions, is equal to the square root of the product.

For example, we can compute $\sqrt{4} \times \sqrt{25}$ in two ways: either as product of the roots

$$\sqrt{4} \times \sqrt{25} = 2 \times 5 = 10$$

or as root of the product

$$\sqrt{4} \times \sqrt{25} = \sqrt{4 \times 25} = \sqrt{100} = 10$$

In the same manner

$$\sqrt{\frac{1}{4}} \times \sqrt{\frac{9}{25}} = \frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$$

$$\sqrt{\frac{1}{4}} \times \sqrt{\frac{9}{25}} = \sqrt{\frac{1}{4} \times \frac{9}{25}} = \sqrt{\frac{9}{100}} = \frac{3}{10}$$

Decimal math

In approximating up to a decimal place, if the digit after the place we want to cut is larger than or equal to 5, then we add 1 more to the digit at which we cut. For example, since 1.4 × 1.7 = 2.38, we take this product up to one decimal place as 2.4.

Like this, is $\sqrt{2} \times \sqrt{3}$ equal to $\sqrt{6}$?

To check this, we must check if the squares of the numbers, 2.4, 2.44, 2.449 and so on, which approximate $\sqrt{2} \times \sqrt{3}$, get closer and closer to 6:

$$2.4^2 = 5.76$$

$$2.44^2 \approx 5.95$$

$$2.449^2 \approx 5.998$$

Since these squares get closer and closer to 6, we have

$$\sqrt{6} = 2.449...$$

and since we also have

$$\sqrt{2} \times \sqrt{3} = 2.449...$$

we can conclude

$$\sqrt{2} \times \sqrt{3} = \sqrt{6}$$

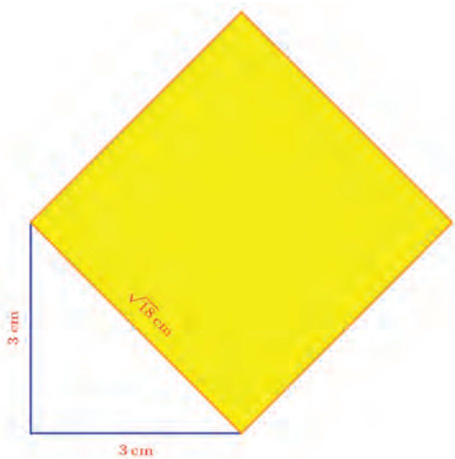
In the same way, we can see that for numbers other than 2 and 3 also, the product of the square roots is the square root of the product. Thus we have this result:

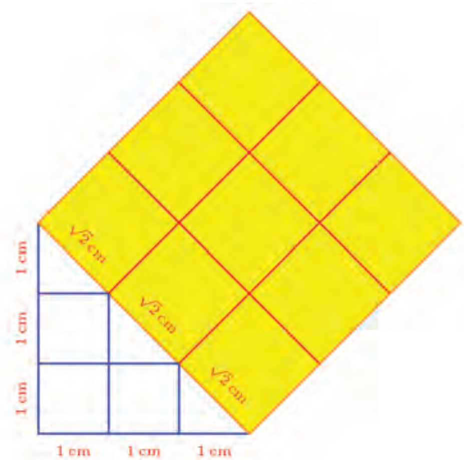
$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$ for all positive numbers x and y

This can be used to simplify certain square roots. As an example, let's look at the hypotenuse of a right triangle with both the perpendicular sides 3 centimetres. By Pythagoras Theorem, the area of the square on the hypotenuse is $3^2 + 3^2 = 18$ square centimetres, so that the length of the hypotenuse is $\sqrt{18}$ centimetres. Now if we write 18 as 9×2, this becomes

$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$$

We can also see this geometrically:





Product and square root

We have seen that the fractions 2.4, 2.44, 2.449 and so on approximate $\sqrt{2} \times \sqrt{3}$ and that the squares of these get closer and closer to 6. Why does this happen?

We got these fractions as approximations to the products

$$1.4 \times 1.7,\ 1.41 \times 1.73,\ 1.414 \times 1.732, ...$$

and so on. The squares of these are

$$(1.4 \times 1.7)^2,\ (1.41 \times 1.73)^2,\ (1.414 \times 1.732)^2, ...$$

and so on. We can write these as products of squares:

$$(1.4 \times 1.7)^2 = 1.4^2 \times 1.7^2$$

$$(1.41 \times 1.73)^2 = 1.41^2 \times 1.73^2$$

$$(1.414 \times 1.732)^2 = 1.414^2 \times 1.732^2$$

In the products on the right side of these products, the numbers 1.4^2, 1.41^2, 1.414^2 and so on get closer and closer to 2 and the numbers 1.7^2, 1.73^2, 1.732^2 and so on get closer and closer to 3. So, their products get closer and closer to 6.

(1) We can make a rectangle using four equilateral triangles of the same size by cutting two of them along their heights and rearranging these pieces and the other two whole triangles:

If the sides of all equilateral triangles are 2 centimetres, what is the perimeter and area of the rectangle ?

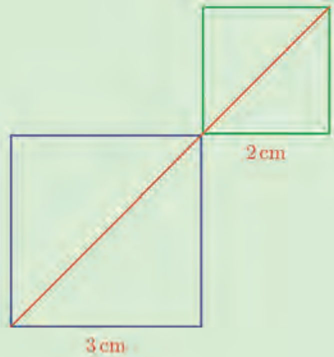
(2) We can make a trapezium by cutting a square and an equilateral triangle with sides twice that of the square, and rearranging the pieces as below:

If the side of the square is 2 centimeters, what is the perimeter and area of the trapezium?

(3) The picture shows the figure formed by joining two squares:

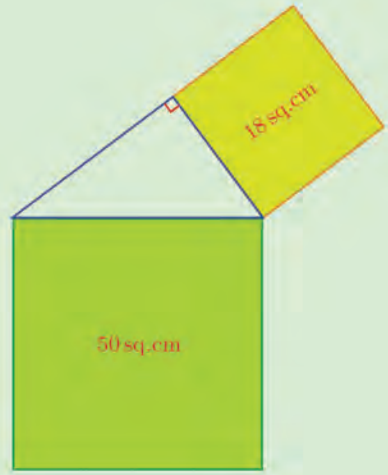
Calculate the length of the bottom side of this figure, correct to a centimetre.

(4) The figure shows two squares with two corners joined:



Find the length of the slanted line.

(5) Calculate the length of the third side of the right triangle in the picture and also its perimeter.



(6) The product of some of the pairs of numbers below are natural numbers or fractions. Find those pairs.

(i) $\sqrt{3}, \sqrt{12}$ (ii) $\sqrt{3}, \sqrt{1.2}$ (iii) $\sqrt{5}, \sqrt{8}$

(iv) $\sqrt{0.5}, \sqrt{8}$ (v) $\sqrt{7\frac{1}{2}}, \sqrt{3\frac{1}{3}}$ (vi) $\sqrt{\frac{2}{5}}, \sqrt{\frac{1}{10}}$

Division

From the product $2 \times 3 = 6$, we get two quotients, $\frac{6}{2} = 3$ and $\frac{6}{3} = 2$

Similarly, we can write the product $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ also as two quotients

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \qquad \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

In general for any x and y which are natural numbers or fractions, the product $x \times y = z$ gives two quotients $\frac{z}{x} = y$ and $\frac{z}{y} = x$

Like this,

For any two positive numbers x and y, the product

$$\sqrt{x} \times \sqrt{y} = \sqrt{z}$$

can be written as the quotients

$$\frac{\sqrt{z}}{\sqrt{x}} = \sqrt{y} \qquad \frac{\sqrt{z}}{\sqrt{y}} = \sqrt{x}$$

Now since $\frac{6}{2} = 3$ and $\frac{6}{3} = 2$ we have

$$\sqrt{\frac{6}{2}} = \sqrt{3}, \qquad \sqrt{\frac{6}{3}} = \sqrt{2}$$

and what did we see earlier ?

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \qquad \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$$

From these two equations, we see that

$$\sqrt{\frac{6}{2}} = \frac{\sqrt{6}}{\sqrt{2}} \qquad \sqrt{\frac{6}{3}} = \frac{\sqrt{6}}{\sqrt{3}}$$

Similarly, we can write

$$\sqrt{3} \times \sqrt{\frac{2}{3}} = \sqrt{3 \times \frac{2}{3}} = \sqrt{2}$$

and from this, the quotient

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

For any two positive numbers x and y

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

Next let's see how such square roots are computed. For example, to compute $\sqrt{\frac{1}{2}}$, we first write

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Then divide 1 by some decimal approximation of $\sqrt{2}$ to get an approximation of $\sqrt{\frac{1}{2}}$.

$$\frac{1}{\sqrt{2}} \approx \frac{1}{1.414} = 0.707$$

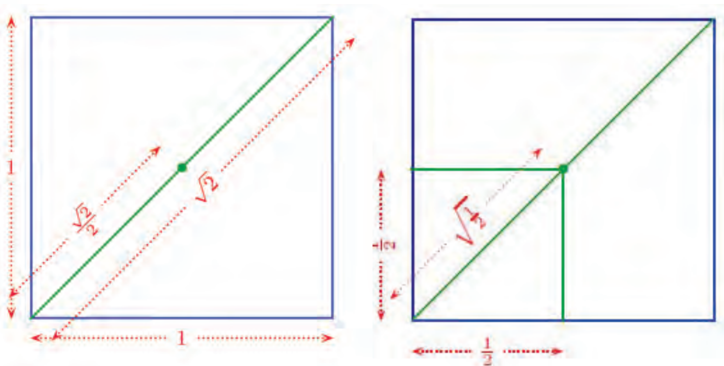
There's a simpler way. Since $\frac{1}{2} = \frac{2}{4}$, we can do the computation like this:

$$\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

Now we can easily calculate

$$\frac{\sqrt{2}}{2} \approx \frac{1.414}{2} = 0.707$$

We can also see geometrically that $\frac{\sqrt{2}}{2} = \sqrt{\frac{1}{2}}$:



Similarly we can compute

$$\frac{1}{\sqrt{3}} = \sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3}$$

In general,

For any positive number x

$$\frac{1}{\sqrt{x}} = \sqrt{\frac{1}{x}} = \sqrt{\frac{x}{x^2}} = \frac{\sqrt{x}}{\sqrt{x^2}} = \frac{\sqrt{x}}{x}$$

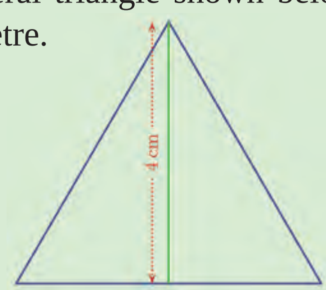
(1) Prove that $(\sqrt{2}+1)(\sqrt{2}-1)=1$.

Using this:

(i) Compute $\frac{1}{\sqrt{2}-1}$ up to two decimal places.

(ii) Compute $\frac{1}{\sqrt{2}+1}$ up to two decimal places.

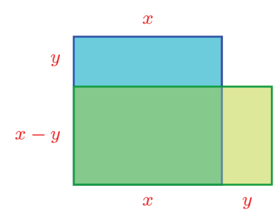
(2) Compute the lengths of the sides of the equilateral triangle shown below, correct to a millimetre.



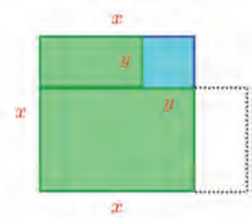
Multiplication identity

Remember the geometry of the identity $(x+y)(x-y) = x^2 - y^2$?

Consider a square with lengths of side x. Suppose that we increase one side by y and decrease another side by y, to form a rectangle:



The area of this rectangle is $(x+y)(x-y)$. We now cut the part of the rectangle jutting out on the right and place it on top:



The area of the green part of the picture is $x^2 - y^2$.

This is equal to the area of the green rectangle we drew first, isn't it ? Area of the rectangle is the product of sides even if the lengths are irrational. So, the identity $(x+y)(x-y) = x^2 - y^2$ is true for irrational numbers also. Similarly we can show that other identities are also true for irrational numbers.

(3) All red triangles in the picture are equilateral and of the same size. What is the ratio of the sides of the outer and inner squares?

(4) Prove that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$ and $\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$. Find other numbers like this.

(5) Among the pairs of numbers given below, find those for which the quotient of the first by the second is a natural number or a fraction.

(i) $\sqrt{72}, \sqrt{2}$ (ii) $\sqrt{27}, \sqrt{3}$ (iii) $\sqrt{125}, \sqrt{50}$

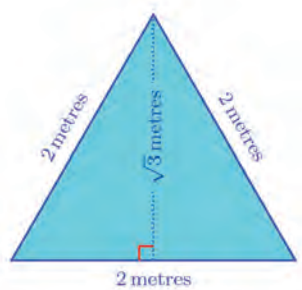
(iv) $\sqrt{10}, \sqrt{2}$ (v) $\sqrt{20}, \sqrt{5}$ (vi) $\sqrt{18}, \sqrt{8}$

Areas of triangles

We defined numbers like $\sqrt{2}$ and $\sqrt{3}$ to denote certain lengths which cannot be expressed as fractions. Later we saw that such numbers are needed to denote the areas of some rectangles also.

If the lengths of the sides of a rectangle are natural numbers or fractions, so would be the area. But the area of triangles with the lengths of all sides natural numbers or fractions may not be such a number.

For example, we have seen that the height of an equilateral triangle with all sides 2 metres is $\sqrt{3}$ metres.

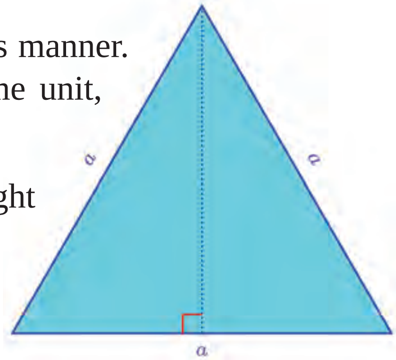


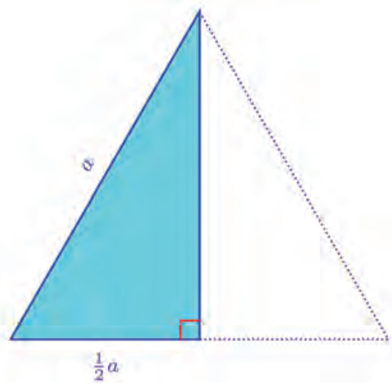
What is its area?

$\frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$ square metres

We can calculate the area of any equilateral triangle in this manner. If we take the length of the sides as a (measured in some unit, centimetre, metre or something else), what is its height?

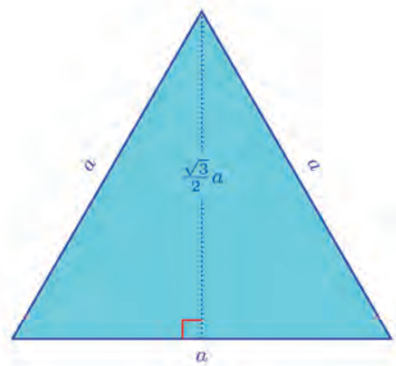
We can write down the hypotenuse and base of the two right triangles within it.





So, the square of the third side is

$$a^2 - \left(\frac{1}{2}a\right)^2 = \frac{3}{4}a^2$$



and from this, the length of side is

$$\sqrt{\frac{3}{4}a^2} = \frac{\sqrt{3}}{2}a$$

This is the height of the equilateral triangle.

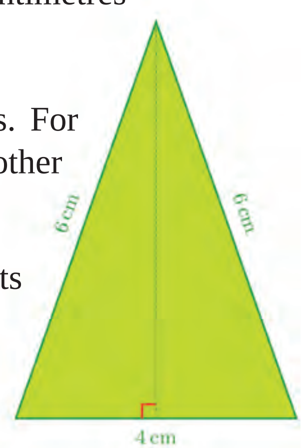
Now we can calculate the area:

$$\text{Area} = \frac{1}{2} \times a \times \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{4}a^2$$

Thus we have this result:

> The altitude of an equilateral triangle is $\sqrt{3}$ times half the side and its area is $\sqrt{3}$ times the square of half the side.

For example, the height of an equilateral triangle with all sides 8 centimetres is $4\sqrt{3}$ centimetres and its area is $16\sqrt{3}$ square centimetres.



We can also calculate the areas of isosceles triangles like this. For example, take the triangle with one side 4 centimetres and the other two 6 centimetres each.

In this triangle also, the perpendicular from the top vertex bisects the base, right ? (The lesson **Equal Triangles** in Class 8)

So, we can compute the square of the height, as we did for equilateral triangles:

$$\text{Square of height} = 6^2 - 2^2 = 32$$

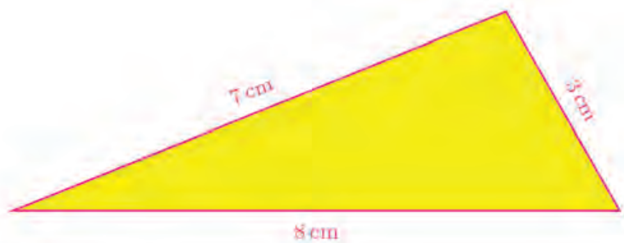
So the height is

$$\sqrt{32} = \sqrt{16 \text{ x } 2} = 4\sqrt{2} \text{ cm}$$

Now we can calculate the area:

$$\text{Area} = \frac{1}{2} \times 4 \times 4\sqrt{2} = 8\sqrt{2} \text{ sq.cm}$$

In much the same way, we can calculate the area of any triangle (requires more computation, that's all). For example, look at this triangle:



First we have to calculate its height. Let's take it as *h* centimetres and the length of one of the pieces it cuts the base as *x* centimetres:

Then from the right triangle on the left, we get

$$x^2 + h^2 = 49$$

and from the right triangle on the right,

$$(8 - x)^2 + h^2 = 9$$

Subtracting the second equation from the first, we get

$$x^2 - (8 - x)^2 = 40$$

We have seen in Class 8 that we can write the difference of two squares as the product of sum and difference. Using this,

$$x^2 - (8 - x)^2 = (x + (8 - x))\,(x - (8 - x))$$

$$= 8(2x - 8)$$

So our equation becomes

$$8(2x - 8) = 40$$

From this, we get $2x - 8 = 5$ and then

$$x = \frac{1}{2}(5 + 8) = 6\frac{1}{2}$$

Now again from the left triangle, we get

$$h^2 = 7^2 - \left(6\frac{1}{2}\right)^2 = 13\frac{1}{2} \times \frac{1}{2}$$
$$= 6\frac{1}{2} + \frac{1}{4} = 6\frac{3}{4}$$

so that

$$h = \sqrt{6\frac{3}{4}} = \sqrt{\frac{27}{4}} = \frac{3}{2}\sqrt{3}$$

Using this, we can calculate the area

$$\text{Area} = \frac{1}{2} \times 8 \times \frac{3}{2}\sqrt{3} = 6\sqrt{3} \text{ sq.cm}$$

In these computations, if we take the lengths of the sides as a, b, c we get a formula to compute the area directly in terms of a, b, c without involving the height:

> If the lengths of the sides of a triangle are a, b, c and we take
>
> $$s = \frac{1}{2}(a + b + c)$$
>
> then the area of the triangle is
>
> $$\sqrt{s(s-a)(s-b)(s-c)}$$

(If you are interested in knowing how this formula is got, see the appendix at the end of this chapter)

This formula was discovered by Heron, a Greek mathematician of the sixth century.

As an example, in the problem we did just now, if we take

$$a = 8 \qquad b = 7 \qquad c = 3$$

then

$$s = \frac{1}{2}(8 + 7 + 3) = 9$$

We next calculate

$$s - a = 1$$
$$s - b = 2$$
$$s - c = 6$$

and put them in Heron's formula to get

$$\text{Area} = \sqrt{9 \times 1 \times 2 \times 6} = \sqrt{9 \times 4 \times 3} = 6\sqrt{3}$$

(1) For each of the lengths below, calculate the area of the equilateral triangle with that as the lengths of the sides:

(i) 10 cm (ii) 5 cm (iii) $\sqrt{3}$ cm

(2) Calculate the area of the regular hexagon with lengths of the sides 6 centimetres.

(3) Calculate the perimeter and area of the equilateral triangle with height 12 centimetres.

(4) Calculate the perimeter and area of the regular hexagon with the distance between parallel sides 6 centimetres.

(5) Calculate the height and area of the triangle with sides 8 centimetres, 6 centimetres, 6 centimetres.

(6) For each of the set of three lengths given below, calculate the area of the triangle with these as the lengths of sides:

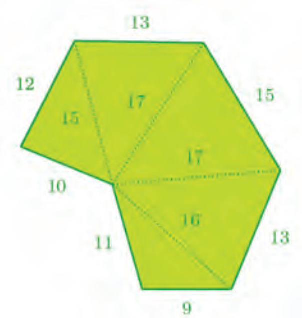
(i) 4 cm, 5 cm, 7cm

(ii) 4cm, 13 cm, 15 cm

(iii) 5 cm, 12 cm, 13 cm

Land area

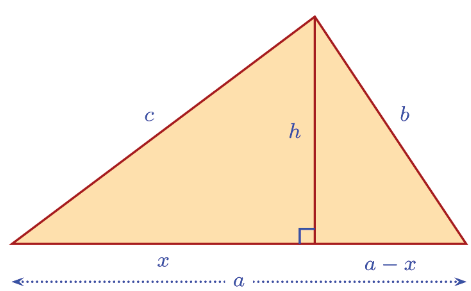
One method of calculating the area of a piece of land, with all boundaries straight, is to divide it into triangles and measure the various lengths:



The the area of each triangle can be computed using Heron's formula and their sum gives the area of the piece of land.

Appendix

Let's see how we get Heron's formula for the area of a triangle. Let us denote the lengths of the sides of the triangle as a, b and c. As we did in an earlier problem, we take the length of the perpendicular from one vertex to the opposite side as h and the length of one of the pieces in which this perpendicular cuts the side as x.



Then from the right triangle on the left we get.

$$x^2 + h^2 = c^2$$

and from that on the right,

$$(a - x)^2 + h^2 = b^2$$

Subtracting the second equation from the first, we get

$$x^2 - (a - x)^2 = c^2 - b^2$$

Next we write the difference of squares $x^2 - (a - x)^2$ as the product of sum and difference:

$$x^2 - (a - x)^2 = (x + (a - x))\,(x - (a - x)) = a(2x - a)$$

So the equation got from the triangle can be written as :

$$a(2x - a) = c^2 - b^2$$

From this we get

$$2x - a = \frac{c^2 - b^2}{a}$$

which gives

$$2x = \frac{c^2 - b^2}{a} + a = \frac{c^2 - b^2 + a^2}{a}$$

Thus we get

$$x = \frac{c^2 - b^2 + a^2}{2a}$$

Now in the equation

$$h^2 = c^2 - x^2 = (c + x)\,(c - x)$$

got from the left triangle, we write x in terms of a, b, c given by the previous equation:

$$h^2 = \left(c + \frac{c^2 - b^2 + a^2}{2a}\right)\left(c - \frac{c^2 - b^2 + a^2}{2a}\right)$$

This can be simplified as below:

$$h^2 = \left(\frac{2ac + (c^2 - b^2 + a^2)}{2a}\right)\left(\frac{2ac - (c^2 - b^2 + a^2)}{2a}\right)$$

The numerator of the first fraction on the right side of the equation can be simplified like this:

$$\begin{aligned} 2ac + (c^2 - b^2 + a^2) &= (a^2 + c^2 + 2ac) - b^2 \\ &= (a + c)^2 - b^2 \\ &= ((a + c) + b)\,((a + c) - b) \\ &= (a + c + b)\,(a + c - b) \end{aligned}$$

And the numerator of the second fraction like this:

$$\begin{aligned}2ac - (c^2 - b^2 + a^2) &= 2ac - c^2 + b^2 - a^2\\ &= b^2 - (a^2 + c^2 - 2ac)\\ &= b^2 - (a - c)^2\\ &= (b + a - c)(b - (a - c))\\ &= (b + a - c)(b - a + c)\end{aligned}$$

Now we can write h^2 using these:

$$h^2 = \frac{(a+c+b)(a+c-b)(b+a-c)(b-a+c)}{4a^2}$$

Now if we write p for the perimeter of the triangle, then

$$p = a + b + c$$

so that

$$\begin{aligned}p - 2a &= (a + b + c) - 2a = (b + c - a)\\ p - 2b &= (a + b + c) - 2b = (a - b + c)\\ p - 2c &= (a + b + c) - 2c = (a + b - c)\end{aligned}$$

So we can write

$$h^2 = \frac{p(p-2b)(p-2c)(p-2a)}{4a^2}$$

Next if we denote half the perimeter (semiperimeter) as s, then $p = 2s$, so that we haves

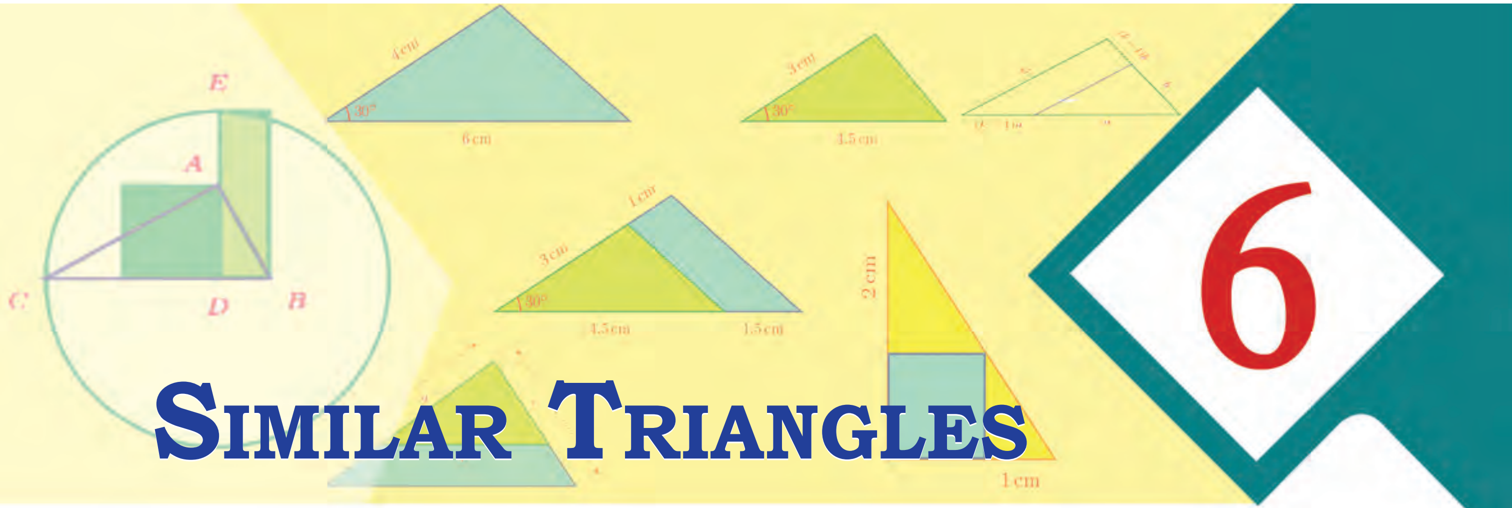
$$\begin{aligned}h^2 &= \frac{2s(2s-2b)(2s-2c)(2s-2a)}{4a^2}\\ &= \frac{2s\times 2(s-b)\times 2(s-c)\times 2(s-a)}{4a^2}\\ &= \frac{4s(s-b)(s-c)(s-a)}{a^2}\end{aligned}$$

From this we get

$$h = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}$$

Finally we can calculate the area:

$$\begin{aligned}\text{Area} &= \frac{1}{2}ah = \frac{1}{2}\times a\times\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}\\ &= \sqrt{s(s-a)(s-b)(s-c)}\end{aligned}$$



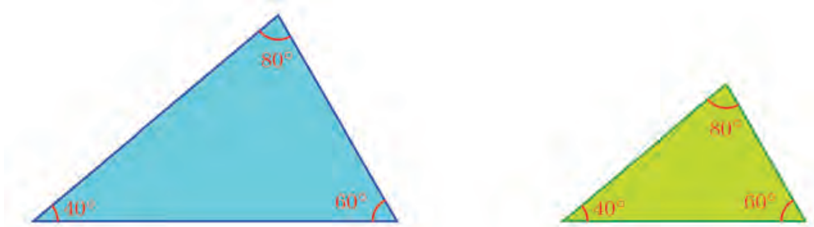
6

Similar Triangles

Angles and sides

We know that if the sides of a triangle are equal to the sides of another triangle, then their angles are also equal. We have also noted that on other hand, even if angles of two triangles are all equal, the sides may not be equal (The lesson, **Equal Triangles** in Class 8). This raise the question: Is there any relation between the sides of two triangles with the same angles ?

To check this, cut out two triangles with the same angles but different sides from a thick sheet of paper. For examples, two triangles like these :

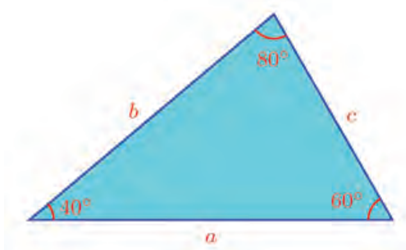


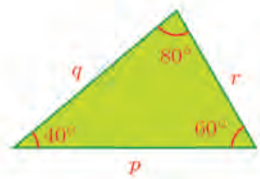
To compare the sides, place the smaller triangle over the larger, with the top corners aligned. Since the angles are equal, the sides making these corners will also be aligned:

Now the bottom sides of both the triangles are equally inclined to the left side. So these sides are parallel.

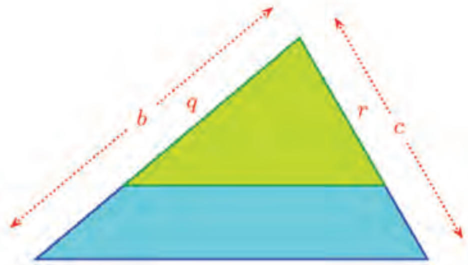
So, the bottom side of the smaller triangle divides the left and right sides of the larger triangle into the same parts of each (The lesson, **Parallel Lines**).

To state this concisely, we denote the lengths of the sides of the triangle by letters as below:





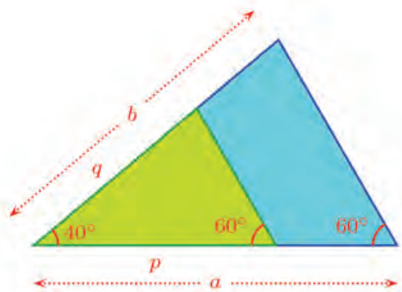
When the smaller triangle is placed over the larger as before, we can mark the lengths as below:



Then the fact (about same parts on either side) found earlier can be written like this:

$$\frac{q}{b} = \frac{r}{c}$$

Suppose we align the left corners instead of the top ones?



In the same manner as above, we can now see that the right sides of the triangles are now parallel, and so the right side of the smaller triangle divides the left and bottom sides of the larger triangle into the same parts of each. That is

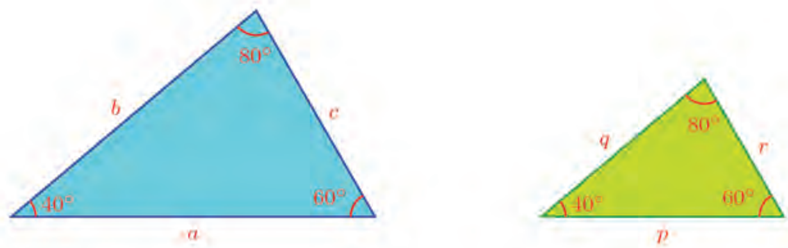
$$\frac{p}{a} = \frac{q}{b}$$

This together with the equation got earlier give

$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c}$$

What does this equation tell us?

First let's look again at what exactly the letters represent:



- a and p are the lengths of the sides opposite the 80° angle
- b and q are the lengths of the sides opposite the 60° angle
- c and r are the lengths of the sides opposite the 40° angle

Next let's see what the fractions in the equation mean:

- The number $\frac{p}{a}$ gives what part of the length a is the length p
- The number $\frac{q}{b}$ gives what part of the length b is the length q
- The number $\frac{r}{c}$ gives what part of the length c is the length r

So what the equation

$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c}$$

means is that all these parts are the same.

That is, if we pair the sides opposite the equal angles of the two triangles as (p, a), (q, b) and (r, c), then the shorter lengths p, q, r are the same part of the longer lengths a, b, c.

We can also say instead that the longer lengths are same times the smaller lengths. This would be true, whatever be the three angles instead of 80°, 60°, 40°.

Thus we have the general result:

In two triangles with the same angles, if we pair the sides opposite equal angles, then the shorter lengths are all the same part of the longer one (or the longer lengths are all same times the shorter).

In any triangle, the shortest side is opposite the smallest angle and the longest side is opposite the largest angle. In other words, the lenghts of the sides follow the same order as the size of the angles.

Draw the triangle ABC in GeoGebra and mark all its angles. Create a slider d with Min = 0. Use the **Segment with Given Length** tool to draw a line of length d times that of AB. For this, give the length of the line as $d*AB$. Next draw triangle DEF with $\angle D = \angle A$ and $\angle E = \angle B$. For this, select the **Angle with Given Size** tool and click on the points E and D in order and in the dialogue window give he measure of the angle as α (the measure of $\angle A$). Similarly, click on D and E in order and give the measure of the angle as ß, with the option clockwise selected. Join the lines DE' and ED' and mark the point of intersection F. Mark the lengths of the sides of both triangles. Are they in the same ratio ? Change the lengths of the sides of triangle ABC and the value of the slider and check

Using this idea, we can shorten the statement of the above result like this:

In triangles with the same angles, the sides, in the order of lengths, are in the same ratio.

We can state this in another manner also. We use the term scaling, in representing one measure as a multiple of another. For example take a 6 centimetre long line and a 4 centimetre long line. The longer length is $1\frac{1}{2}$ times the other, while the shorter one is $\frac{2}{3}$ of the other. We say that the longer line is got by scaling (up) the shorter by a factor of $1\frac{1}{2}$ and the shorter is got by scaling (down) the longer by a factor of $\frac{2}{3}$. We can also say that the scale factor from the shorter to the longer is $1\frac{1}{2}$, while the scale factor from the longer to the shorter is $\frac{2}{3}$.

We've seen that if the triangle with lengths of sides a, b, c and p, q, r have the same angles, then p, q, r are the same times a, b, c. This means the change from the sides of the first triangle to the second has the same scale factor. If this scale factor is taken as k, then the relation between the lengths of the sides can be written:

$$a = kp,\ b = kq,\ c = kr$$

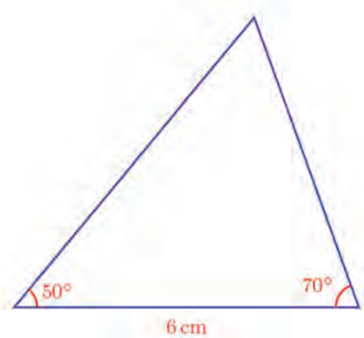
So, the general result can be stated like this:

In triangles with the same angles, the sides opposite equal angles are scaled by the same factor.

Now look at this problem:

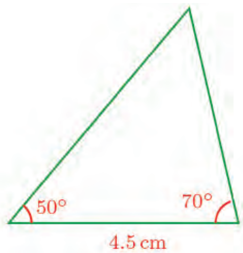
We want to draw a smaller triangle with all its sides $\frac{3}{4}$ of the sides of this triangle:

The bottom side of the triangle we want should be 4.5 centimetres long. What about the other sides?



Should we first draw the large triangle, measure its other two sides, and draw the smaller triangle with sides $\frac{3}{4}$ of this?

We need only draw a line 4.5 centimetre long and draw the same angles as those of the larger triangle, right ?



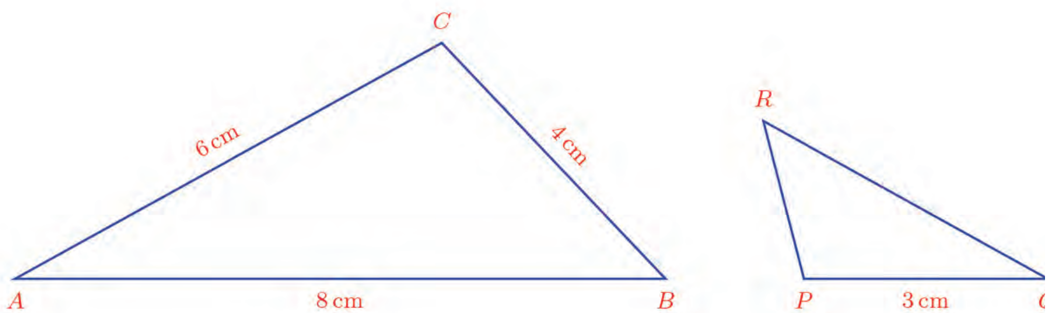
Since the angles are equal, the other sides will also be $\frac{3}{4}$ part of those of the larger triangle, by the general result we've seen.

Triangle speciality

If two triangles have the same angles, their sides are in the same ratio. This is a speciality not shared by other polygons. For example, look at these pictures:

In both the rectangle and the square, all angles are right. The lengths of the left sides of both are in the same (ratio 1 : 1); so are the right sides. But the top sides of the rectangle and the square are of different lengths; the bottom sides are also of different lengths.

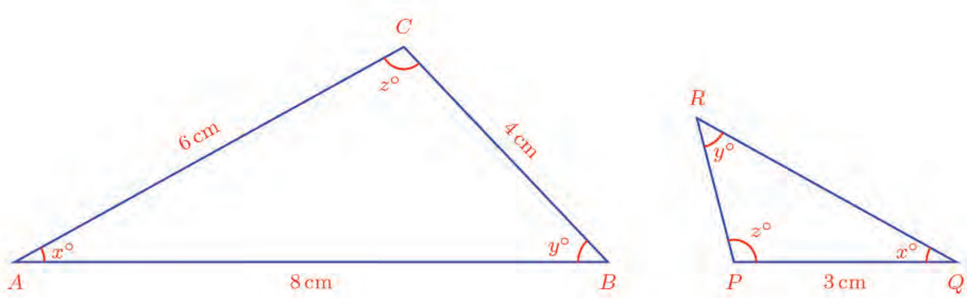
Look at another problem:



$$\angle P = \angle C \qquad \angle Q = \angle A \qquad \angle R = \angle B$$

How do we compute the lengths of the other two sides of the triangle PQR ?

First let's mark the equal angles, taking their measures as $x°$, $y°$, $z°$.



Next we write the pairs of sides opposite equal angles:

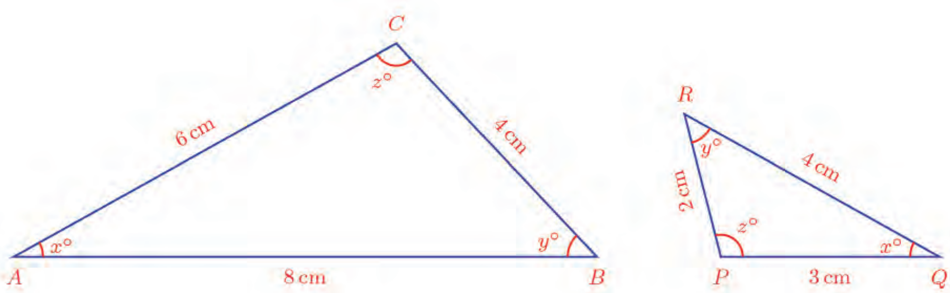
x	BC	PR
y	AC	PQ
z	AB	QR

We know the lengths of all sides of the larger triangle and the length of one side of the smaller:

x	$BC = 4$	PR
y	$AC = 6$	$PQ = 3$
z	$AB = 8$	QR

We see that for the sides opposite the $y°$ angle, the shorter side is half of the longer. So the sides opposite other angles must be scaled by the same factor:

x	$BC = 4$	$PR = 2$
y	$AC = 6$	$PQ = 3$
z	$AB = 8$	$QR = 4$



If we can compare the lengths of the sides of the smaller triangle just by sight, we can compute the lengths without writing the angles:

	Shortest	Medium	Longest
ΔABC	$BC = 4$	$AC = 6$	$AB = 8$
ΔPQR	PR	$PQ = 3$	QR

From this we can see that the sides of the smaller triangle are half those of the larger and using this, calculate the lengths of the other two sides.

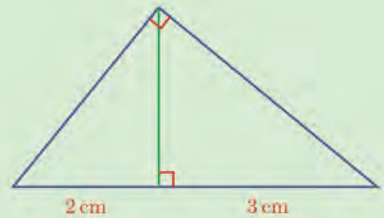
(1) One side of a triangle is 8 centimetres and the two angles on it are 60° and 70°. Draw the triangle with lengths of sides $1\frac{1}{2}$ times that of this triangle and with the same angles.

(2) In a right triangle, the perpendicular from the square corner to the hypotenuse divides it into pieces 2 centimetres and 3 centimetres long:

(i) Prove that the two small right triangles formed by this perpendicular have the same angles

(ii) Taking the height of the perpendicular as h, prove that $\frac{h}{2} = \frac{3}{h}$.

(iii) Calculate the lengths of the perpendicular sides of the original large triangle.



(iv) Prove that if the length of the perpendicular from the square corner of a right triangle to the hypotenuse is h and it divides the hypotenuse into pieces of length a and b, then $h^2 = ab$.

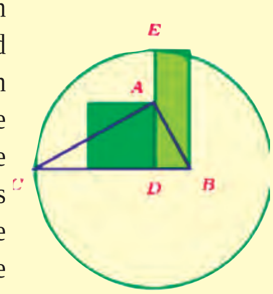
(3) At the two ends of a horizontal line, angles of the same size are drawn and two points on these slanted lines are joined :

(i) Prove that the horizontal line (blue) and the slanted line (red) cut each other into parts in the same ratio.

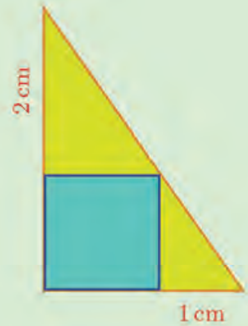
(ii) Prove that the slanted lines (green) at the ends of the horizontal line are also in the same ratio.

(iii) Explain how this idea can be used to divide a 6 centimetre long line in the ratio 3 : 4.

Draw right triangle ABC in GeoGebra and draw the perpendicular from the square corner A to the hypotenuse BC. Mark the point D where it meets the hypotenuse. Draw the circle with centre at D and passing through C and mark the point E where the perpendicular meets the circle. Draw the square with one side AD and the rectangle with sides BD and DE. Mark their areas. Aren't they equal ? Change the positions of the right triangle and check.



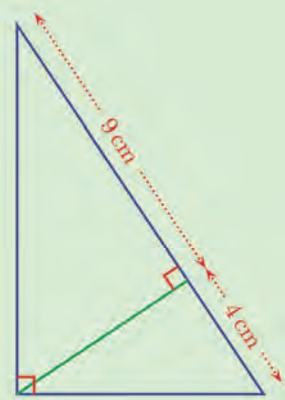
(4) The picture below shows a square sharing one corner with a right triangle and the other three corners on the sides of this triangle:



(i) Calculate the length of the sides of the square.

(ii) What is the length of the sides of such a square drawn within a triangle of sides 3 centimetres 4 centimetres, 5 centimetres?

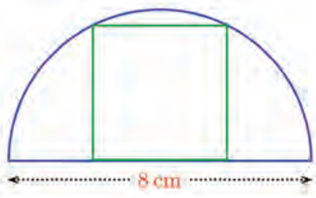
(5) Calculate the area of the largest right triangle in the picture below:



(6) Two poles of heights 3 metres and 2 metres are erected upright on the ground and ropes are stretched from the top of each to the foot of the other:

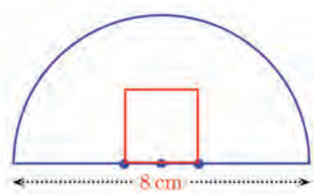
Semicircle and square

See this picture:

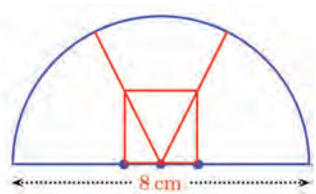


How do we inscribe a square within a semicircle like this?

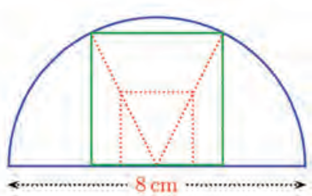
First mark two points on the diameter of the semicircle, equidistant from the centre. Draw a square on this part of the diameter:



Join the centre of the circle to the top corners of this square and extend these lines to meet the semicircle.



Join the points where these lines meet the semicircle. Draw perpendiculars to the diameter from these points:



Can you prove that what we get thus is infact a square?

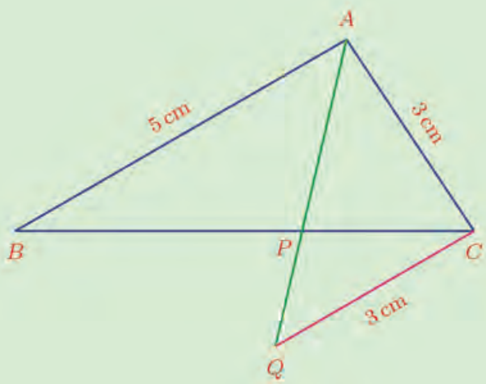
(i) At what height above the ground do the ropes cross each other ?

(ii) Prove that this height would be the same whatever the distance between the poles.

(iii) Denoting the heights of the poles as a, b and the height of the point of crossing above the ground as h, find the relation between a, b and h.

Draw a horizontal line and mark points C and D on it. Draw perpendiculars to the first line through these. Mark a point E on the perpendicular through C and a point F on the perpendicular through D. Draw the lines ED and FC and mark their point of intersection G. Now change the positions of C and D and check the height of G above the horizontal line. Right click on G and select the Trace On. What is the path of G as the distance between C and D changes?

(7) In the picture below, AP is the bisector of $\angle A$ of triangle ABC:



Draw triangle ABC in GeoGebra and draw the bisector of $\angle C$. Mark the point D where it meets AB. Compare the ratio of the lengths of the sides AC and BC and the ratio of the lengths AD and BD. Typing AC/BC in the **Input Bar** gives the number $\frac{AC}{BC}$. The number $\frac{AD}{BD}$ found like this.

(i) Prove that angles of the triangles ABP and CPQ are the same.

(ii) Calculate $\frac{BP}{PC}$

(iii) Prove that in any triangle, the bisector of an angle divides the opposite side in the ratio of the sides containing the angle.

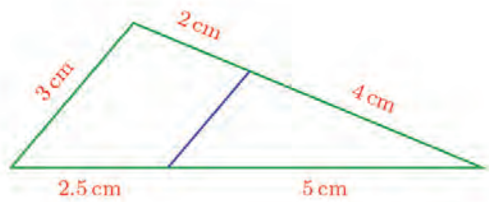
Sides and angles

We saw that if the angles of two triangles are the same, then their sides are scaled by the same factor. This raises the reverse question: if all sides of a triangle are scaled (stretched or shrunk) by the same factor, would the angles remain the same ?

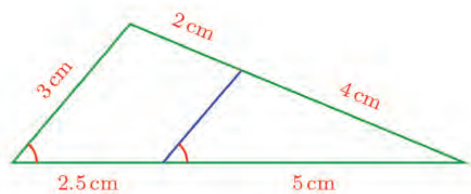
See this picture:

The sides of the larger triangle are all one and a half times those of the smaller triangle. Are the angles of the triangles the same ?

To check this, we first mark the lengths of two sides of the small triangle on the large triangle and join these points as shown below:



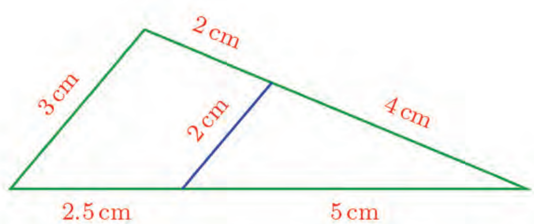
This line divides the bottom side and the right side of the large triangle in the same ratio 1 : 2. So, it is parallel to the left side (The section **Triangle division** in the lesson, **Parallel Lines**). This means their inclinations with the bottom side are the same.



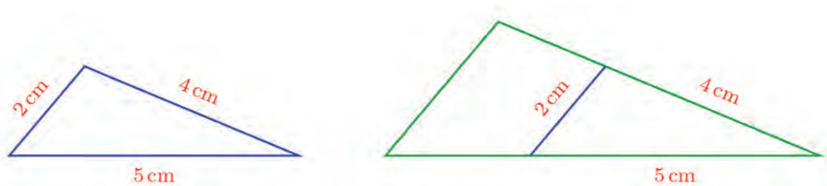
So, if just look at the large triangle and the small triangle within it (let's ignore the small triangle outside for the time being), we see that they have the same angles.

By the general result seen earlier, the sides of these two triangles are scaled by the same factor.

The bottom side of the small triangle is $\frac{2}{3}$ of the bottom side of the large triangle. The right sides are also scaled by the same factor. So, the left sides of the triangles must also be scaled by this factor. Thus we can calculate the third side of the small triangle.



Now let's look again at the small triangle outside the large triangle, which we had kept aside:



The small triangles within and out have sides of the same length and so their angles are also equal (The lesson, **Equal Triangles** in Class 8).

And we have seen that the angles of the large triangle are the same as those of the small triangle within it.

So what do we get?

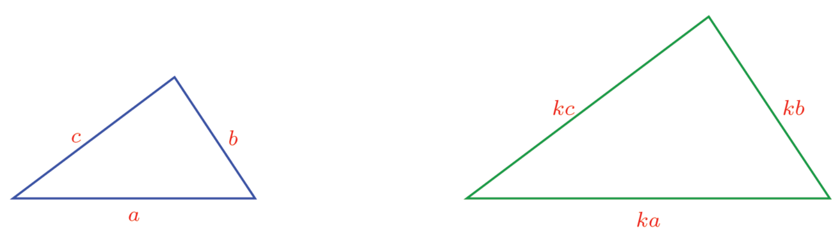
The angles of the small and large triangles we started with are the same.

Even if we change the lengths of the sides and the scale factor used in this example, we can show that the angles of the two triangles are the same, using the same arguments as above.

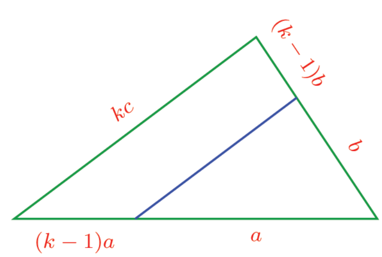
For those who need greater precision, we can do the same thing in a general setting using algebra.

Take two triangles with lengths of sides of one scaled by the same factor in the other. That is, the lengths of the sides of one triangle are multiples of those of the other by the same number.

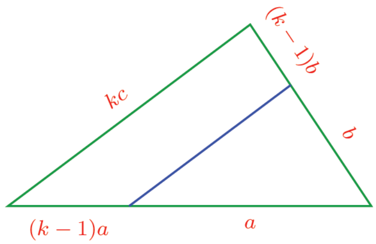
So, if we take the lengths of the sides of the small triangle as *a*, *b*, *c*, then those of the large triangle can be taken *ka*, *kb*, *kc*.



As we did in the example, we mark the lengths of two sides of the small triangle on two sides of the large triangle and join these points, as shown below:

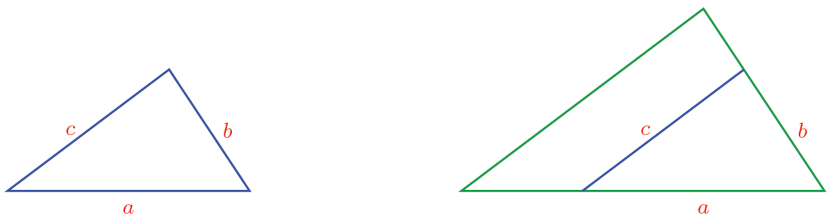


This line divides the bottom and right sides of the large triangle in the same ratio $(k-1):1$. So, this line is parallel to the left side. So, the large triangle and the small one inside have the same angles.



This means their sides must be scaled by the same factor. The bottom side of the small triangle within is $\frac{1}{k}$ part that of the bottom side of the large triangle (same is the case with right sides). So, the left sides also must be scaled by the same factor:

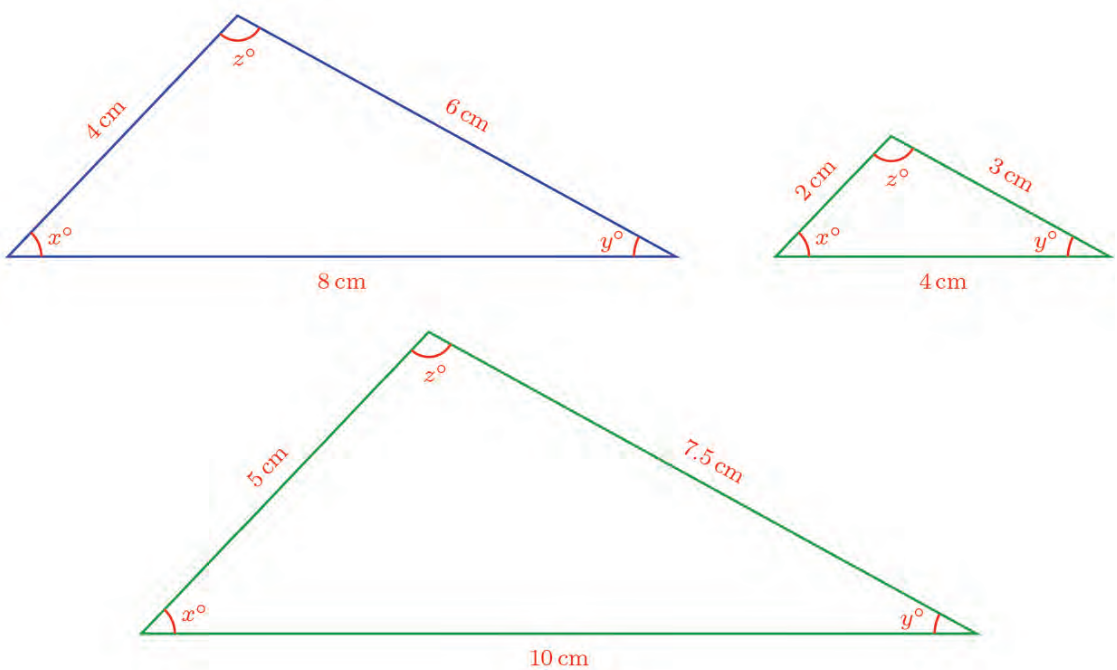
Now as in the example, we look at the small triangles within and outside the large triangle:



Since the lengths of the sides of these triangles are the same, the angles must also be the same; and we have seen that the angles of the large triangle are the same as those of the small triangle within. Thus the small and large triangles we started with have the same angles.

If two triangles have their sides scaled by the same factor, then their angles are the same.

So to enlarge or shrink a triangle without altering the angles, we need not measure the angles. We need only scale the sides by the same factor:



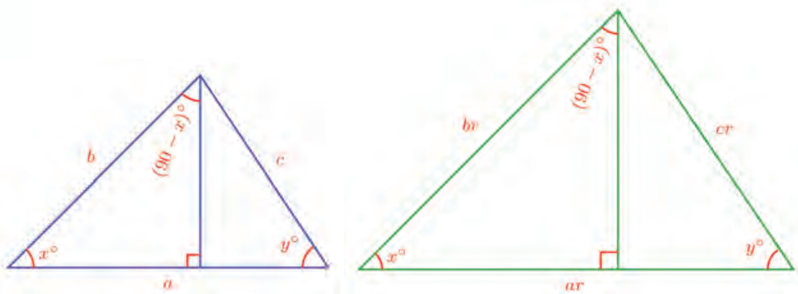
Let's look at a problem using this idea. We can easily see that if all sides of a triangle are scaled by the same factor, then the perimeter also is scaled by the same factor. (Try it!)

What about the areas?

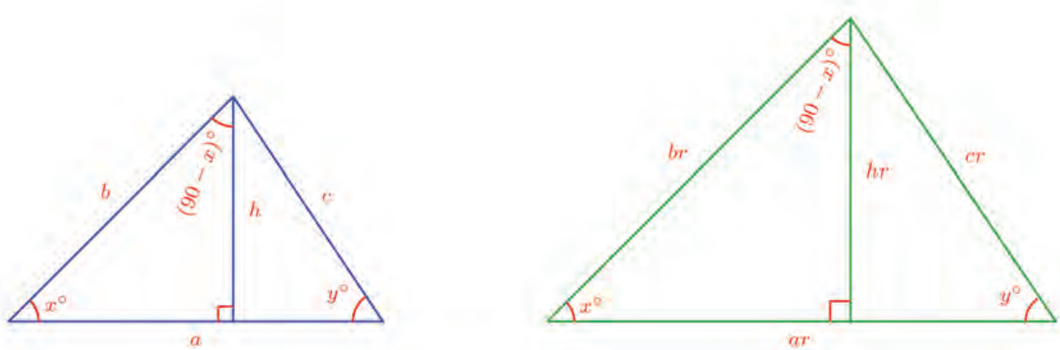
To see it, let's draw two such triangles. As we have seen just now, they have the same angles. To compare the areas, let's draw perpendiculars from two vertices with the same angles:

Draw triangle ABC in GeoGebra with names of the sides a, b, c. Create a slider k with Min=0. Draw line DE with length ka. With D and E as centres, draw circles of radii ka and kb and mark one of their points of intersection as F. Draw triangle DEF. Mark the angles of both triangles. Aren't they equal? Change the value of the slider and positions of the vertices of the first triangle. What do you see?

Mark the perimeter and area of the two triangles. What is the scale factor of the perimeters? And of the areas?



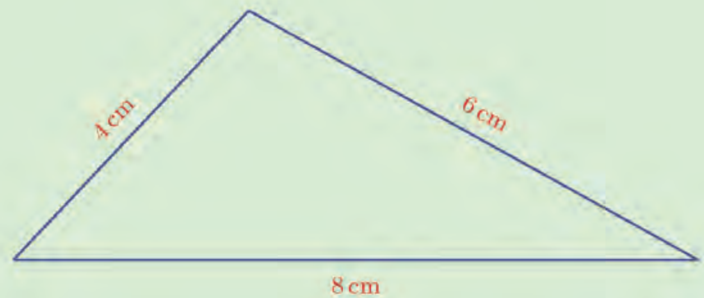
Focus on the two right triangles on the left. Both have the same angles, $x°$, $90°$, $(90 - x)°$. So, their sides are scaled by the same factor. The hypotenuse of the blue triangle is b and that of the green is br. So, if the length of the vertical side of the blue triangle is taken as h, then the length of the vertical side of the green triangle is hr.



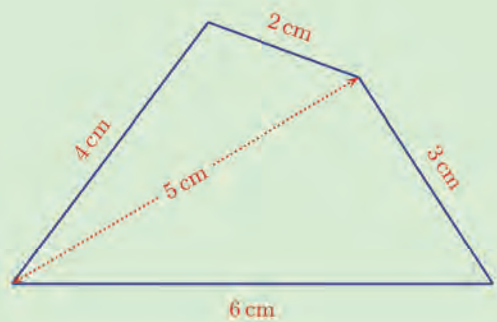
Now we can compute the areas of the whole triangles. The area of the blue triangle is $\frac{1}{2}ah$; that of the green triangle is $\frac{1}{2}ahr^2$.

Thus the scale factor of area is the square of the scale factor of the sides.

(1) Draw the triangle with angles the same as those of the triangle shown below, and sides scaled by $1\frac{1}{4}$.



(2) See the picture of the quadrilateral.



(i) Draw the quadrilateral with the same angles as this and sides scaled by a factor of $1\frac{1}{2}$.

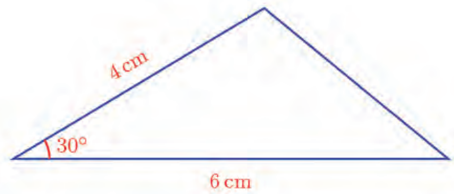
(ii) Draw a quadrilateral with angles different from this and sides scaled by a factor of $1\frac{1}{2}$.

(3) The area of a triangle is 6 square centimetres. What is the area of the triangle with lengths of sides four times those of this? What about the one with lengths of sides half of this?

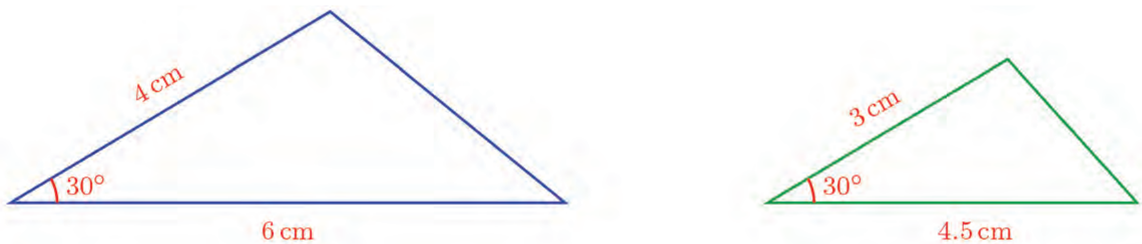
Third way

We saw in the first section how we can scale a triangle up or down without altering its angles, if we know the length of one side and two angles on it. Scale the known side by the required factor and draw the same angles at each end; the other sides would be scaled by the same factor. In the second section we saw how this could be done, if the lengths of the three sides are known instead: just scale all sides by the required factor; the angles would be the same.

Now suppose we want to do this on a triangle with lengths of two sides and the included angle known. For example see this picture. We want to scale it down by a factor of $\frac{3}{4}$ without altering the angles.

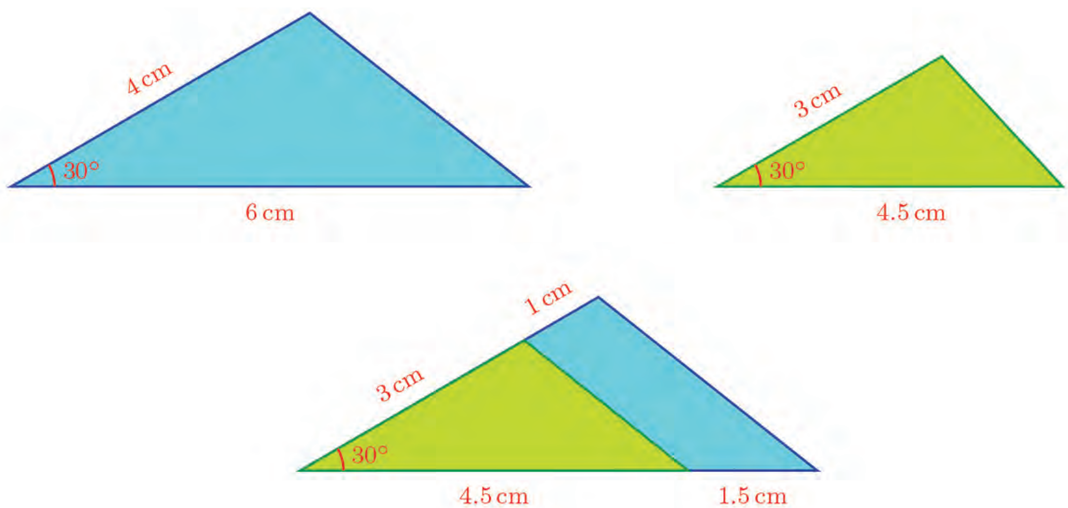


We can draw a triangle with two sides $\frac{3}{4}^{\text{th}}$ of 6 centimetres and 4 centimetres joined at an angle of 30°:

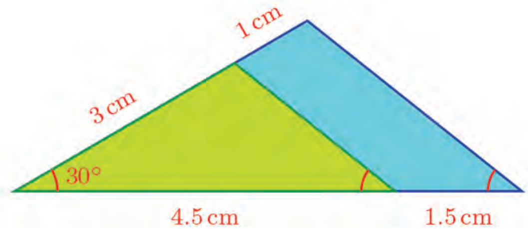


But we don't know whether the third side is also scaled down by $\frac{3}{4}$ and whether the other two angles of the triangles are the same.

To check this, cut out these triangles from a thick sheet of paper and place the smaller over the larger with the left corners aligned, as we did in the first section. Since the angles are equal, the sides meeting at this corner would also be aligned:



Now the right side of the green triangle divides the left and bottom sides of the blue triangle in the same ratio. So, it is parallel to the right side of the blue triangle. This means the right sides of the triangles are equally inclined to the bottom side.

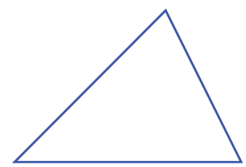


Thus we see that the two triangles have the same angles. So, their sides must be scaled by the same factor. Thus we also see that the third side of the small triangles is also $\frac{3}{4}$ of the third side of the large triangle.

Now even if the measures and the scale factor are different, we can reach the same conclusion using the above arguments.

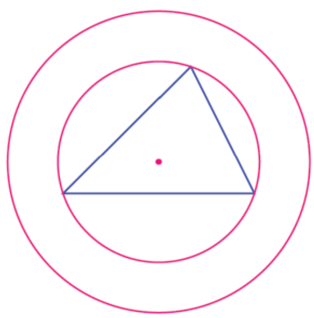
If two triangles have two of the sides scaled by the same factor and the included angles equal, then the third sides are also scaled by the same factor and the other two angles are also equal.

Using this result, we can scale a triangle without measuring any side or angle. For example, draw a triangle anyway you like.

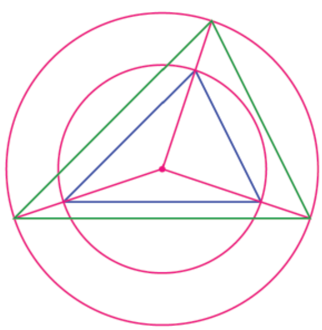


Let's see how the sides can be stretched one and a half times.

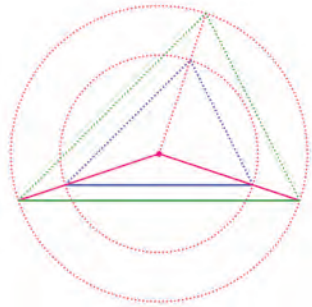
For this draw the circumcircle of the triangle and with the same centre, draw another circle of one and a half times the radius:



Now join the centre of the circles to the vertices of the triangle and extend them to meet the larger circle. Join these points to draw a larger triangle



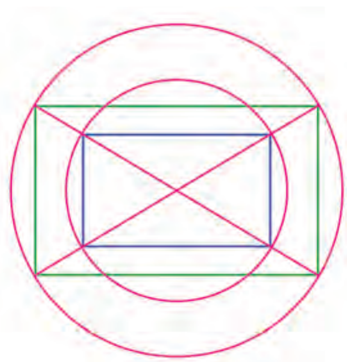
To see that the sides of the larger (green) triangle are one and a half times the sides of the smaller (blue) triangle, first focus on the triangles highlighted in the picture below:



The left and right sides of the larger triangle in this are one and a half times the left and right sides of the smaller triangle (why?) And the angle between them is the same for both the triangles. So, the bottom sides are also scaled by the same factor.

This means the bottom side of the green triangle drawn earlier is one and a half times the bottom side of the blue triangle drawn first. In the same way, we can see that the other sides of these triangles are also scaled by the same factor.

In a similar manner, we can enlarge or shrink a rectangle also:



In GeoGebra, create a slider *a* with Min=0. Draw triangle *ABC* and find its circumcentre *D*. (Use **Circle through 3 Points** tool to draw the circumcircle; select the **Midpoint** or **Centre** tool and click on the circle to get its centre). With *D* as centre draw a circle of radius 'a' times the circumradius (Give *a*AD* as radius). Using the **Ray** tool draw lines starting at *D* and passing through the vertices of the triangle. Mark the points where these meet the new circle and join them to make a triangle. Find the relation between the lengths of the sides of the two triangles.

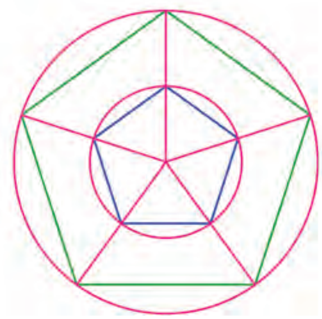
We cannot use this method for all quadrilaterals. For example, we cannot draw a circle through the four vertices of a parallelogram which is not a rectangle.

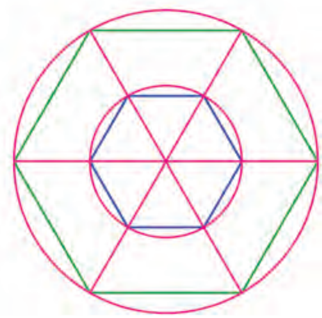
Right triangles

Just because two sides of a triangle are equal to two sides of another, the triangles may not be equal. Either the third sides must also be equal or the included angles must be equal. But two sides of a right triangle determine the third side, by the Pythagoras Theorem, so that two right triangles with the two sides with the same length would be equal.

In the same manner, two triangles with just two sides scaled by the same factor may not be similar; either the third sides must also be scaled by the same factor or the included angles must be equal. But in two right triangles, if two sides are scaled by the same factor, the third side also is scaled by this factor itself, and so the triangles are similar.

However, we can use this method to scale any regular polygon.





We summarize all the results we have discussed:

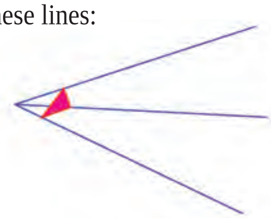
If two triangles have any one of the following relations, they also have the other two:

- Have the same angles
- Have all sides scaled by the same factor
- Have two sides scaled by the same factor and the included angle equal

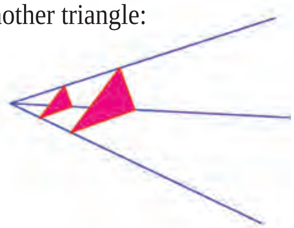
Triangles having any of these relations between them are said to be *similar.*

Projection

Join the vertices of a triangle to a point outside and extend these lines:

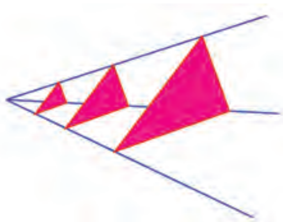


Draw a line parallel to the left side of the triangle, a little to the right of the triangle. Through the point where this meets the top blue line, draw another line parallel to the right side of the triangle. Through the point where it meets the middle blue line, draw a line parallel to the bottom side of the triangle. Now we get another triangle:



The two triangles have the same angles (why?) and so their sides are stretched by the same factor.

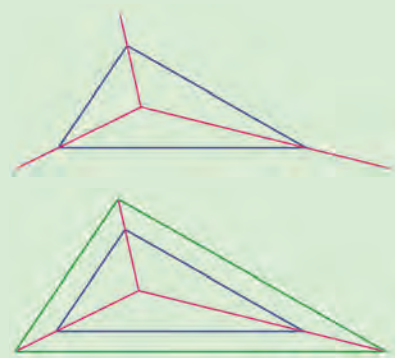
We can draw as many triangles as we want like this.



(1) Prove that if the perpendicular sides of two right triangles are scaled by the same factor, then the hypotenuses are also scaled by the same factor.

(2) Prove that if any two sides of two right triangles are scaled by the same factor, then the third side also is scaled by the same factor.

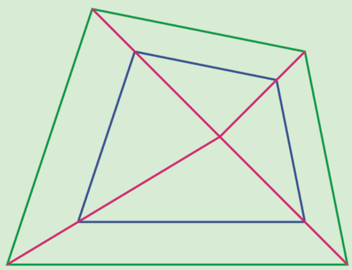
(3) Draw a triangle and mark a point inside it. Join this point to the vertices and extend each of them by half its original length. Join the end points of these lines to form another triangle;



Prove that the sides of the larger triangle are one and a half times the sides of the original triangle.

Draw triangle *ABC* in GeoGebra and mark a point *D* outside it. Using the **Ray** tool, join lines from *D*, passing through the vertices of the triangle. Mark a point *E* on the line *DA* and draw the line through *E*, parallel to *AB*. Mark the point *F* where this line meets *DB*. Similarly draw the line through *E*, parallel to *AC* and mark the point *G* where this line meets *DC*. Draw triangle *EFG*. Mark the angles of triangles *ABC* and *EFG*. Aren't they the same? Change the position of *E*. Doesn't the size of triangle *EFG* change? Are the sides of these two triangles scaled by the same factor?

(4) The vertices of a quadrilateral are joined to a point inside it and these line are extended by the same scale factor. The ends of these lines are joined to form another quadrilateral:



(i) Prove that the sides of the two quadrilaterals also are scaled by the same factor.

(ii) Prove that the angles of the two quadrilaterals are the same.

We can use the **Enlarge from Point** (**Dilate from Point**) tool in GeoGebra to draw similar triangles. Create a slider *a* with Min = 0. Draw a triangle and mark a point inside or outside it. Select the **Enlarge from Point** tool and click first on the triangle and then on the point. In the dialogue window, give *a* as the scale factor. We get another triangle similar to the first. Change the position of the point and the value of *a* and check. Instead of triangles, we can draw figures similar to any other figure in this way.

Project

Check whether the altitudes, medians and angle bisectors of similar triangles are scaled by the same factor as the sides.

7

Negative Numbers

Measures and numbers

We've seen in class 8, how we use negative numbers to denote temperatures below zero. We set zero degree celsius, written 0°C, as the temperature at which water freezes to solid ice. To denote temperatures colder than this, we have to use notations like −1°C, −20.5°C.

We've also seen how negative numbers are used to denote points in some games and as scores in some exams. We also noted some general results on calculations with such numbers.

For example, look at this question:

If temperature dropped 7°C from 3°C, what would it be?

To answer this we introduced the operation

$$3 - 7 = -4$$

and through other examples like this, came up with the general law for subtracting a small number from a larger number.

Through such examples of operations needed in different contexts, we formulated three general laws about operations with negative numbers:

For any two positive numbers x, y

(i) If $x < y$, then $x - y = -(y - x)$

(ii) $-x + y = y - x$

(iii) $-x - y = -(x + y)$

Using these ideas, can't you do the following?

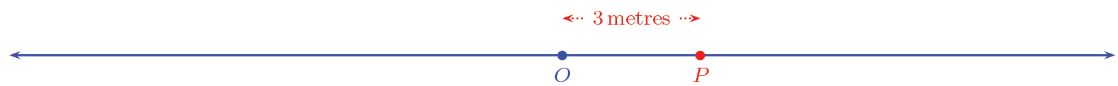
(i) $6 - 8$ (ii) $-6 + 8$ (iii) $-6 - 8$

(iv) $2\frac{1}{2} - 3\frac{1}{2}$ (v) $-2\frac{1}{2} + 3\frac{1}{2}$ (vi) $-2\frac{1}{2} - 3\frac{1}{2}$

We next look at another context in which negative numbers are used.

Position and number

Imagine a point moving along a straight line. To indicate its position, we can specify its distance (measured in some unit) from a fixed point:



The picture shows the position of the moving point *P* at a distance of 3 metres to the right of the fixed point *O*.

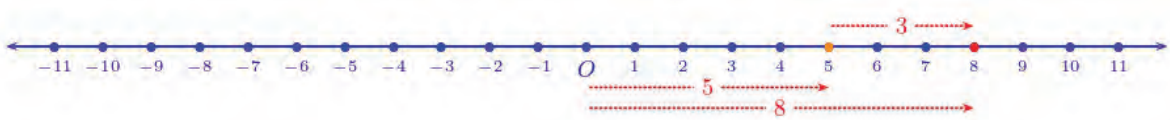
P can also be at 3 metres to the left of *O*.

So to know the exact position of *P*, distance alone will not do; we'll have to specify the direction also.

One way to overcome this, is to specify distances to the left as negative numbers. Once we have decided on a unit of length (such metre or centimetre), we can denote all positions, except *O*, on the line using positive and negative numbers.

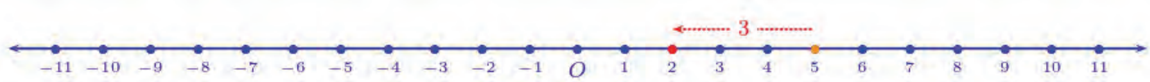
The position of the fixed point *O* can be denoted by 0.

Now suppose that the moving point shifts from the position 5 to the position 8. We can say that the displacement (change in position) is 8 – 5 = 3:



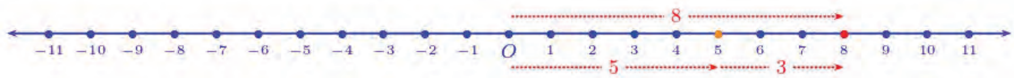
On the other hand, if we say that the point shifts 3 places from 5, can we say that its current position is 8 ?

It's correct if the change in position is to the right; what if it is to the left ?

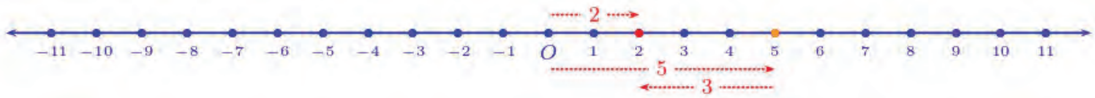


So it's not enough to know the amount of displacement, we should also know whether it is to the right or left.

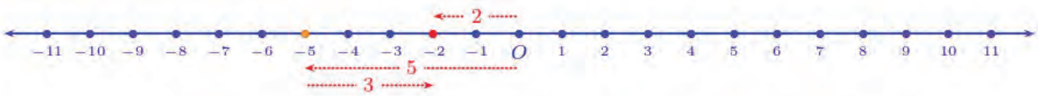
That is, if the point shifts 3 places to the right from 5, then it is at 8.



If the shift is to the left, the position is 2:



What if the point shifts 3 places to the right from –5 instead?



Is there a general method by which we can compute the final position, if we know the initial (first) position and the displacement with direction ?

For this, let's tabulate the various possibilities. First we consider only displacements to the right:

Initial Position	Displacement	Final Position
5	3 right	8
3	5 right	8
−5	3 right	−2
−3	5 right	

Write the last number also by drawing a picture (or by visualizing in the mind).

Let's write just the numbers for the displacements:

Initial Position	Displacement	Final Position
5	3	8
3	5	8
−5	3	−2
−3	5	2

Is there any relation between the numbers in each row?

We can see at a glance that in the first two rows, the third number is the sum of the first two numbers.

What about the next line?

We've seen earlier that

$$-5 + 3 = 3 - 5 = -2$$

And the last line?

We've also seen that

$$-3 + 5 = 5 - 3 = 2$$

So in every row of the table, the last number is the sum of the first two numbers:

Initial Position	Displacement	Final Position
5	3	8 = 5 + 3
3	5	8 = 3 + 5
–5	3	–2 = –5 + 3
–3	5	2 = –3 + 5

Now let's look at displacements to the left:

Initial Position	Displacement	Final Position
5	3 left	2
3	5 left	–2
–5	3 left	–8
–3	5 left	–8

As in the case of positions, let's write displacements to the left also as negative numbers:

Initial Position	Displacement	Final Position
5	–3	2
3	–5	–2
–5	–3	–8
–3	–5	–8

As in the first table, is the last number in each row, the sum of the first two in this also ?

To check this for the first row here, we must compute the sum 5 + (–3). But we haven't seen such a sum before.

Let's think like this: for two positive numbers, even if we change the order of summation, the sum doesn't change; for example

$$5 + 3 = 3 + 5 \qquad \frac{1}{2} + \frac{1}{4} = \frac{1}{4} + \frac{1}{2} \qquad \sqrt{2} + \sqrt{3} = \sqrt{3} + \sqrt{2}$$

So here also, we can take

$$5 + (-3) = -3 + 5$$

and in this, we've already seen that

$$-3 + 5 = 5 - 3 = 2$$

Thus we define

$$5 + (-3) = 5 - 3 = 2$$

Then in the first line of this table also, we'll get the last number as the sum of the first two numbers.

By the same token, if we take

$$3 + (-5) = -5 + 3 = 3 - 5 = -2$$

then the relation between the numbers in the second row would also be the same:

Initial Position	Displacement	Final Position
5	–3	2 = 5 + (–3)
3	–5	–2 = 3 + (–5)
–5	–3	–8
–3	–5	–8

What about the remaining two rows ?

In those also, we should give meaning to the sum of the first two numbers. In the first two rows, we wrote the addition of a negative number as a difference, such as

$$5 + (-3) = 5 - 3 = 2$$

$$3 + (-5) = 3 - 5 = -2$$

How about doing this for the last two rows as well ?

$$(-5) + (-3) = -5 - 3$$

$$(-3) + (-5) = -3 - 5$$

We've already seen subtractions as in the right of these equations:

$$-5 - 3 = -(5 + 3) = -8$$

$$-3 - 5 = -(3 + 5) = -8$$

Thus if we define

$$(-5) + (-3) = -5 - 3 = -8$$

$$(-3) + (-5) = -3 - 5 = -8$$

then the relation between the numbers in the last two rows would also be the same as that in the first two rows:

Initial Position	Displacement	Final Position
5	–3	2 = 5 + (–3)
3	–5	–2 = 3 + (–5)
–5	–3	–8 = –5 + (–3)
–3	–5	–8 = –3 + (–5)

Now let's combine our two tables for displacements to the right and left:

Initial Position x	Displacement y	Final Position $x + y$
5	3	8 = 5 + 3
3	5	8 = 3 + 5
–5	3	–2 = (–5) + 3
–3	5	2 = (–3) + 5
5	–3	2 = 5 + (–3)
3	–5	–2 = 3 + (–5)
–5	–3	–8 = (–5) + (–3)
–3	–5	–8 = (–3) + (–5)

Here's another table like this, giving different initial positions and displacements:

Initial Position	Displacement	Final Position
7	3 right	
3	7 right	
–7	3 right	
–3	7 right	
7	3 left	
3	7 left	
–7	3 left	
–3	7 left	

Draw pictures to find the final positions and write in the table. Then make another table with the displacements as just numbers, positive or negative and check if in each row, the last number is the sum of the first two numbers.

Now to make the last number in each row, the sum of the first two numbers, we gave meaning to some new operations, such as

$$3 + (-5) = 3 - 5 = -2$$

$$5 + (-3) = 5 - 3 = 2$$

$$(-5) + (-3) = -5 - 3 = -8$$

$$(-3) + (-5) = -3 - 5 = -8$$

We note that in all these equations, the left side is the addition of a negative number; and it is defined as subtraction of a positive number in the right side.

We use algebra to make this more precise:

For any number x and any positive number y,

$$x + (-y) = x - y$$

Using this, we see that in our position problem, the displacement added to the initial position gives the final position, whatever be the kind of numbers these are.

That is, if we denote the initial position by x, displacement by y and final position by z, we can use the single equation

$$z = x + y$$

to find the final position, whatever be the kind of numbers x, y, z are.

Here we could write the relation connecting x, y, z in a single equation only because we could take them as positive or negative numbers.

Just think how many equations we would need, if we use only positive numbers with left or right directions specified in words.

One fact, many equations

What if we try to write the connection between positions and displacement in algebra, without using negative numbers? We can denote the distances from O by x and z and the distance between them by y. Since these are all to be positive numbers as distances, we will also have to specify the directions left or right. So we will have to write the relation between them in different ways depending on the situation:

For x and y in the same direction
$z = x + y$ in the same direction

For x and y in different directions

if $x > y$

then $z = x - y$ in the direction of x

if $x < y$

then $z = y - x$ in the direction of y

(1) Complete the table below:

x	y	$x + y$
6	−10	
−6	10	
−6	−10	
−6	6	
6	−6	

(2) Find two pairs of numbers x and y satisfying each of the following conditions:

(i) x positive, y negative with, $x + y = 1$

(ii) x negative, y positive with, $x + y = 1$

(iii) x positive, y negative with $x + y = -1$

(iv) x negative, y positive with $x + y = -1$

(3) Complete the table below:

x	y	z	$(x + y) + z$	$x + (y + z)$
2	4	−5		
2	−4	5		
−2	4	−5		
2	−4	−5		
−2	4	5		
−2	−4	5		
−2	−4	−5		

Displacement

We can have a different problem on positions of a moving point: to move from the position 4 to the position 7, how many places should the point move?

It is not enough to say 3 places; if the shift is towards the left from 4, it would reach 1, right ? So to move from 4 to 7, we must say 3 places to the right.

How about moving from 7 to 4 ?

Again 3 places, but now to the left.

How many places to shift to move from 4 to −7 ?

Let's draw a picture:

A different difference

An easy way to subtract 9 from 17 is to subtract 9 from 10 and add 7:

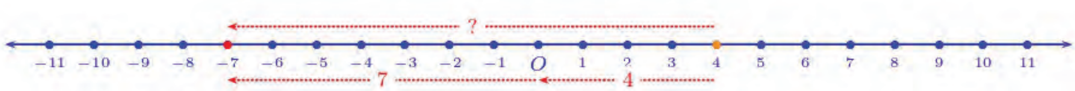
$$\begin{aligned} 17 - 9 &= (10 + 7) - 9 \\ &= (10 - 9) + 7 \\ &= 1 + 7 = 8 \end{aligned}$$

We can also do using addition of negative numbers.

$$\begin{aligned} 17 - 9 &= (10 + 7) - 9 \\ &= 10 + (7 - 9) \\ &= 10 + (-2) \\ &= 10 - 2 = 8 \end{aligned}$$

In the same way,

$$\begin{aligned} 57 - 29 &= (50 - 20) + (7 - 9) \\ &= 30 + (-2) \\ &= 30 - 2 = 28 \end{aligned}$$



Let's calculate the displacements for various initial and final positions, to the left and right, and make a table as before. First we write displacements as positive numbers with the qualifications left or right:

Initial Position	Final Position	Displacement
4	7	3 right
7	4	3 left
4	−7	11 left
−7	4	
−4	7	
7	−4	
−4	−7	
−7	−4	

Next let's write displacements also as just numbers, with those to the left negative:

Initial Position	Final Position	Displacement
4	7	3
7	4	−3
4	−7	−11
−7	4	11
−4	7	11
7	−4	−11
−4	−7	−3
−7	−4	3

Now let's see in this table also, what the relation between the three numbers in each row is.

The first displacement was actually calculated as $7 - 4 = 3$.

Can we do this in the second row also?

$$4 - 7 = -3$$

So in this row also, we get the third number by subtracting the first from the second. What about the next row?

We've seen that

$$-7 - 4 = -(7 + 4) = -11$$

Thus this calculation works here also.

To check whether this holds in the fourth row, we have to say what the operation $4-(-7)$ means.

Let's think like this. Subtraction of positive numbers can be described in many ways. One way of interpreting $10-6$ is, finding what should be added to 6 to make 10; then from $6+4=10$, we get $10-6=4$

Similarly we can think of the operation $4-(-7)$ as finding what should be added to -7 to make 4. This can be done in two steps. 7 added to -7 makes 0. To get 4 from 0, we must add 4 more. Thus we have to add $7+4=11$ to get 4 from -7.

According to this interpretation,

$$4-(-7)=7+4=11$$

It is better to write this as

$$4-(-7)=4+7=11$$

for easy recall later.

Thus in the fourth row also, the third number is got by subtracting the first from the second. Thinking along the same lines, we can also see that

$$7-(-4)=7+4=11$$

This makes the fifth row of the table also conform to our rule.

As for the sixth row we have

$$-4-7=-(4+7)=-11$$

as seen from earlier definitions.

Let's write our computations so far in the table:

Initial Position	Final Position	Displacement
4	7	$3 = 7 - 4$
7	4	$-3 = 4 - 7$
4	-7	$-11 = -7 - 4$
-7	4	$11 = 4 - (-7)$
-4	7	$11 = 7 - (-4)$
7	-4	$-11 = -4 - 7$
-4	-7	-3
-7	-4	3

To continue this for the next line, we should say what (–7) – (−4) means. Recall that we defined

$$7 - (-4) = 7 + 4$$

earlier in this problem itself. Similarly, we define

$$(-7) - (-4) = -7 + 4$$

We've already seen

$$-7 + 4 = 4 - 7 = -3$$

Thus we have

$$(-7) - (-4) = -7 + 4 = -3$$

which makes the seventh row also right.

In the same way we define

$$(-4) - (-7) = -4 + 7 = 3$$

and then everything is fine with the last row also and thus with the whole table:

Initial Position x	Final Position y	Displacement $y - x$
4	7	3 = 7 – 4
7	4	–3 = 4 – 7
4	–7	–11 = – 7 – 4
–7	4	11 = 4 – (–7)
–4	7	11 = 7 – (– 4)
7	–4	–11 = – 4 – 7
–4	–7	–3 = (–7) – (–4)
–7	–4	3 = –4 – (–7)

Now to make this right, we've made some new definitions of subtraction:

$$4 - (-7) = 4 + 7 = 11$$

$$7 - (-4) = 7 + 4 = 11$$

and

$$(-7) - (-4) = -7 + 4 = -3$$

$$(-4) - (-7) = -4 + 7 = 3$$

In general, we defined the subtraction of a negative number as removing the negative sign and adding.

This we can write in algebra as below:

For any number x and any positive number y

$$x - (-y) = x + y$$

Suppose we take $x = 0$ and $y = 3$ in this equation. We get

$$0 - (-3) = 0 + 3 = 3$$

We have $0 - 3 = -3$. Similarly if we write $0 - (-3)$ simply as $-(-3)$, then using the above equation we get,

$$-(-3) = 0 - (-3) = 0 + 3 = 3$$

This is a new definition, that the meaning of the negative of negative of a positive number is the number itself.

For any positive number x.

$$-(-x) = 0 - (-x) = 0 + x = x$$

Using this, we can extend the first equation we gave at the beginning of this lesson. This is the equation we mean:

For any two positive numbers x and y, if $x < y$, then $x - y = -(y - x)$

What if $x > y$ in this equation ?

For example, let's take $x = 10$ and $y = 3$.

$$x - y = 10 - 3 = 7$$

$$y - x = 3 - 10 = -7$$

$$-(y - x) = -(-7) = 7$$

Here also, we get $x - y = -(y - x)$

In general, if $x < y$ in this equation, then $y - x$ is a positive number and $x - y$ is its negative

If $x > y$, then $y - x$ is negative and $x - y$ is its negative

Meaning of negative

We introduced negative numbers through situations requiring numbers less than zero. Later we saw that they could be used to distinguish between opposite directions. And we made some new definitions of addition and subtraction to facilitate this interpretation. Continuing with these, when we defined the negative of negative, the process of taking the negative becomes a mathematical operation.

Since we have defined the negative of the negative of a positive number as that number itself, the equation we are considering holds in the second case also.

So we can remove the condition in the equation and generalize it:

For any numbers x and y

$$x - y = -(y - x)$$

(1) Take different numbers, positive and negative, as x, y, z and calculate $x - (y - z)$ and $(x - y) + z$. Are both the same number in all cases ?

(2) Find two pairs of numbers x and y, satisfying each of the conditions below:

(i) x positive, y negative, $x - y = 1$

(ii) x negative, y positive, $x - y = -1$

(3) (i) Take four consecutive natural numbers or their negatives as a, b, c, d and calculate $a - b - c + d$

(ii) Explain using algebra why it is zero for all such numbers.

(iii) What do we get if we calculate $a + b - c - d$ instead of $a - b - c + d$?

(iv) What about $a - b + c - d$?

Time and speed

We saw how we can write the relations between the positions and displacement of a point moving along a straight line as general algebraic equations using positive and negative numbers, even if the positions and displacement are in different directions.

For the point to be displaced, it must move; and to describe motion in terms of numbers, we must take into account not only displacement, but time also.

If we are told that an object travelled 50 metres in 5 seconds, we get an idea of its speed. If the speed is also known to be steady with no increase or decrease during the journey, we can say that the speed is 10 metres per second. We can also calculate that if it continues like this, it will cover 100 metres in 10 seconds (We can also remember that the fastest time so far in 100 metre sprint is 9.58 seconds).

For a point moving along a line with steady speed in either direction, if we know its speed and the time of travel, then we can calculate the distance travelled. For example if the speed is 10 metres per second (written 10 m/sec), then in 4 seconds it travels a distance of $10 \times 4 = 40$ metres. But to fix its final position, we must know the direction of travel.

As before, let us imagine a point travelling along a straight line, moving either left or right with respect to a fixed point *O*. As usual, we mark positions on the line using distances from the point *O*, positive numbers for positions to the right of *O* and negative numbers for positions to the left of *O*. Let's suppose that the point starts from *O*. Also, fix the unit of distances as metre, unit of time as second and the unit of speed as metre per second.

If the point starts from *O* and travels to the right at the speed of 10 m/s for 3 seconds, it would reach the position $3 \times 10 = 30$.

If the direction of travel is to the left ? The position is $-(3 \times 10) = -30$

Speed math

The fastest time in 100 metre sprint is currently 9.58 seconds, set by Usain Bolt of Jamaica in 2009 at Berlin.

He didn't run at a steady speed throughout this time. The time he took to cover every 20 metres has been recorded:

Distance Matres	Time Seconds
0 – 20	2.89
20 – 40	1.75
40 – 60	1.67
60 – 80	1.61
80 – 100	1.66

So to calculate the position, if the travel is to the right, we multiply the speed by time; if it is to the left we take the negative of this product.

To state this in algebra, we denote the time by t, speed by v, and position by s. Then

$$s = tv \text{ if moving to the right}$$

$$s = -(tv) \text{ if moving to the left}$$

To combine them into a single equation, we can try denoting speeds to the right by positive numbers and that to the left by negative numbers.

For example if the travel is to the right at 10 m/s, we take $v = 10$ and if it is to the left, $v = -10$.

But then, there's a problem: for a point starting from O and moving to the left at 10 m/s for 3 seconds, if we want to calculate the position as a product, we will have to define what we mean by $3 \times (-10)$.

We can think about this like this: for positive numbers, multiplication is computing so many times something. For example,

$$3 \times 10 = 3 \text{ times } 10 = 10 + 10 + 10 = 30$$

How about defining $3 \times (-10)$ also like this?

$$3 \times (-10) = (-10) + (-10) + (-10) = -30$$

This definition of multiplying a negative number by a natural number we can extend to all numbers.

So denoting the time by t, the speed by v and the position by s, we can say the relation between them by the general equation, $s = tv$

Now let's change the context a bit. Suppose a point is moving along the line at a steady speed of 10 m/s and we start observing it only from the time it reaches O.

If the journey is towards from left to right, at 2 seconds after we start watching, its position would be at 20.

What about the position at 2 seconds before we start watching?

It needs 2 more seconds to reach O. So it's at a distance of 20 metres to the left of O, that is at the position -20.

And if the journey is from right to left?

Let's calculate the positions for different directions of travel at times before and after we start observing, and make a table. To write down the positions using an actual picture or mental image, we first write speed and positions using the qualifications right and left:

Time	Speed	Position
2 seconds after	10 right	20 right
2 seconds after	10 left	20 left
2 seconds before	10 right	20 left
2 seconds before	10 left	20 right

Now let's write position and speed as only numbers, positive or negative:

Time	Speed	Position
2 seconds after	10	20
2 seconds after	–10	–20
2 seconds before	10	–20
2 seconds before	–10	20

To remove the qualifications before and after for time, we can write time after observation as positive and times before observation as negative:

Time	Speed	Position
2	10	20
2	–10	–20
–2	10	–20
–2	–10	20

In the first row of this table, the last number is got by multiplying the second number by the first.

In the second row also, according our new definition of multiplication, we have $2 \times (-10) = -20$. What about the third line ?

What is the meaning of $(-2) \times 10$?

We think like this. For positive numbers, changing the order of numbers in a multiplication doesn't change the product. For example

$$5 \times 3 = 3 \times 5 \qquad \frac{1}{2} \times \frac{1}{4} = \frac{1}{4} \times \frac{1}{2} \qquad \sqrt{2} \times \sqrt{3} = \sqrt{3} \times \sqrt{2}$$

So here also, we define $(-2) \times 10 = 10 \times (-2)$

And we've already seen that

$$10 \times (-2) = -20$$

Thus if we define

$$(-2) \times 10 = 10 \times (-2) = -20$$

Then the same relation, as in the first two rows, holds for the numbers in the third row of the table also.

What about the last row?

How do define the meaning of $(-2) \times (-10)$?

We have to define this as 20, if we want the relation between the numbers in each row.

We can give a purely mathematical justification for such a definition like this.

To multiply the sum of two positive numbers by a positive number, we need only multiply each and add. For example,

$$(5 + 2) \times 10 = (5 \times 10) + (2 \times 10)$$

In algebraic language,

$$(x + y)z = xz + yz \text{ for any positive numbers } x, y, z$$

Let's see what we'll have to do to make this true for negative numbers also. For example, let's take $x = 5$, $y = -2$, $z = -10$ and calculate the two sides of the above equation $(x + y)z = xz + yz$ separately.

$$(x + y)z = (5+ (-2)) \times (-10) = 3 \times (-10) = -30$$

$$xz + yz = (5 \times (-10)) + ((-2) \times (-10)) = -50 + ((-2) \times (-10))$$

If this products are to be equal, what should be $(-2) \times (-10)$?

The first product is -30 and the second product is some number added to -50

What is the number to be added to –50 to make it –30 ?

So, if we want the equation $(x + y)z = xz + yz$ to be true for $x = 5$, $y = -2$, $z = -10$ we'll have to define $(-2) \times (-10)$ as 20.

If we define like this, we get the last number in each row of our table as the product of the first two:

Time t	**Speed** v	**Position** tv
2	10	$20 = 2 \times 10$
2	–10	$-20 = 2 \times (-10)$
–2	10	$-20 = (-2) \times 10$
–2	–10	$20 = (-2) \times (-10)$

Thus we can give the relation between the time t, the speed v and the positions s can be given by the single equation

$$s = tv$$

The new definitions used to get this are these

For any two positive numbers x and y

$$(-x)y = x(-y) = -(xy)$$

$$(-x)\,(-y) = xy$$

(1) Take different numbers, positive and negative, as x, y, z and calculate $(x + y)\,z$ and $x\,z + yz$. Check if the equation

$(x + y)\,z = x\,z + yz$ holds in all cases.

(2) Prove that the equation $(x+y)\,(u+v) = xu + xv + yu + yv$ holds if x, y, u, v are replaced by $-x, -y, -u, -v$.

Oscillation

What is $(-1) \times (-1)$? By the definition of multiplying negative numbers, it is 1. What about $(-1) \times (-1) \times (-1)$?

The product of the first two –1's is 1. When this is multiplied by –1 again, we get –1. We can write these as

$$(-1)^2 = 1$$

$$(-1)^3 = -1$$

Compute some more powers of –1. Can you find a general pattern?

If the exponent is an odd number, then the power is –1; if exponent is even, the power is 1.

The algebraic form of this is

$$(-1)^{2n-1} = -1$$

$$(-1)^{2n} = 1$$

for any natural number n.

Now take different natural numbers as n and compute $1 + (-1)^n$.

What about $\frac{1}{2}(1 + (-1)^n)$?

(3) In each of the equations below, find y when x is the given number:

(i) $y = x^2$, $x = -1$ (ii) $y = x^2 + 3x + 2$, $x = -1$

(iii) $y = x^2 + 3x + 2$, $x = -2$ (iv) $y = (x + 1)(x + 2)$, $x = -1$

(v) $y = (x + 1)(x + 2)$, $x = -2$

(4) In the equation $y = x^2 + 4x + 4$, take different numbers, positive and negative, as x and calculate y. Why is y positive or zero in all cases?

(5) Natural numbers, their negatives and zero can be together called integers.

(i) How many pairs of integers (x, y) can you find, satisfying the equation $x^2 + y^2 = 25$

(ii) How many pairs of integers (x, y) can you find satisfying $x^2 - y^2 = 25$?

(6) $1 \times 2 \times 3 \times 4 \times 5 = 120$, what is $(-1) \times (-2) \times (-3) \times (-4) \times (-5)$?

(7) What is $(1 \times 2 \times 3 \times 4 \times 5) + [(-1) \times (-2) \times (-3) \times (-4) \times (-5)]$?

Negative division

For positive numbers, division is defined in terms of multiplication. For example, the meaning of $6 \div 2$ is finding that number which on multiplication by 2 gives 6,

Since $2 \times 3 = 6$, write

$$6 \div 2 = 3$$

Since $3 \times 2 = 6$ also, we have

$$6 \div 3 = 2$$

According to this, what is $(-6) \div 2$?

Since $2 \times (-3) = -6$, we have

$$-6 \div 2 = (-3)$$

Also, since $(-3) \times 2 = -6$, we have

$$-6 \div (-3) = 2$$

Similarly, since $(-3) \times (-4) = 12$, we have

$$12 \div (-3) = -4$$

and

$$12 \div (-4) = -3$$

We can calculate other divisions using negative numbers in the same way.

In algebra, we usually write $x \div y$ as $\frac{x}{y}$. So, the general rules of negative division can be stated like this:

For any positive numbers x and y

$$\frac{-x}{y} = \frac{x}{-y} = -\left(\frac{x}{y}\right)$$

$$\frac{-x}{-y} = \frac{x}{y}$$

Unification of equations

An object thrown upwards reaches a certain height and starts falling down. This can be mathematically described.

If thrown straight up, the speed decreases at the rate of 9.8 m/s. Continuously decreasing thus, when the speed becomes zero, it starts falling down. Throughout the fall, the speed increases at the rate of 9.8 m/s.

Suppose it is thrown up with a speed of 49 m/s. At 5 seconds, the speed becomes $49 - (5 \times 9.8) = 0$. So after 5 seconds, the journey is downwards with increasing speed. So, at 7 seconds, the speed is $0 + (2 \times 9.8) = 19.6$ m/s

So we need two equations to describe this journey algebraically:

$$v = 49 - 9.8t, \text{ if } t < 5$$

$$v = 9.8t - 49, \text{ if } t > 5$$

If we denote the speed upwards by positive numbers and the speed downwards by negative numbers, we can describe the motion using a single equation:

$$v = 49 - 9.8t$$

(1) Calculate y when x is taken as $2, -2, \frac{1}{2}, -\frac{1}{2}$ in the equation

$$y = \frac{1}{x}$$

(2) Calculate y when $x = -2$ and $x = -\frac{1}{2}$ in the equation

$$y = \frac{1}{x-1} + \frac{1}{x+1}$$

(3) Calculate z when x and y are taken as the numbers below in the equation $z = \frac{x}{y} - \frac{y}{x}$

(i) $x = 10, y = -5$ (ii) $x = -10, y = 5$ (iii) $x = -10, y = -5$

Notes

Notes

CONSTITUTION OF INDIA

Part IV A

FUNDAMENTAL DUTIES OF CITIZENS

ARTICLE 51 A

Fundamental Duties- It shall be the duty of every citizen of India:

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wild life and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievements;

(k) who is a parent or guardian to provide opportunities for education to his child or, as the case may be, ward between age of six and fourteen years.

CHILDREN'S RIGHTS

Dear Children,

Wouldn't you like to know about your rights? Awareness about your rights will inspire and motivate you to ensure your protection and participation, thereby making social justice a reality. You may know that a commission for child rights is functioning in our state called the ***Kerala State Commission for Protection of Child Rights.***

Let's see what your rights are:

- Right to freedom of speech and expression.
- Right to life and liberty.
- Right to maximum survival and development.
- Right to be respected and accepted regardless of caste, creed and colour.
- Right to protection and care against physical, mental and sexual abuse.
- Right to participation.
- Protection from child labour and hazardous work.
- Protection against child marriage.
- Right to know one's culture and live accordingly.
- Protection against neglect.
- Right to free and compulsory education.
- Right to learn, rest and leisure.
- Right to parental and societal care, and protection.

Major Responsibilities

- Protect school and public facilities.
- Observe punctuality in learning and activities of the school.
- Accept and respect school authorities, teachers, parents and fellow students.
- Readiness to accept and respect others regardless of caste, creed or colour.

Contact Address:

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Child Helpline - 1098, Crime Stopper - 1090, Nirbhaya - 1800 425 1400

Kerala Police Helpline - 0471 - 3243000/44000/45000

Online ***R. T. E Monitoring : www.nireekshana.org.in***