Standard X

MATHEMATICS

Part-2

Government of Kerala
Department of General Education

State Council of Educational Research and Training (SCERT), Kerala
2019

NT-889-1-MATHS-10-E-VOL.2

THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha.
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give respect to my parents, teachers and all elders and treat everyone with courtesy.

I pledge my devotion to my country and my people. In their well-being and prosperity alone lies my happiness.

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Dear children,

Measurements and relations between them form an important part of mathematics. Because of this, physical and social sciences require mathematics to present such quantitative relations. Mathematics also has an ideal aspect in which measurements are seen as pure numbers and objects as geometrical shapes. Relations between numbers then grow into algebra, and cause-effect relationships of physical objects develop into logical connections between ideas. Thus mathematical theorems are formed. These in turn lead to more effective practical applications. Here we present the basic lessons in mathematical theory and applications.

In the present age, tedious computations and complex drawings are done using computers. On the other hand, knowledge of mathematics is essential for the effective use of computers. We have indicated this two-way interaction between mathematics and computers at many places in this book. We have included examples of using the dynamical geometry program **GeoGebra** and the computer language **Python**. More material on these are made available through the Samagra portal and through QR codes.

With love and regards
Dr. J. Prasad
Director, SCERT

CONSTITUTION OF INDIA
Part IV A
FUNDAMENTAL DUTIES OF CITIZENS

ARTICLE 51 A

Fundamental Duties- It shall be the duty of every citizen of India:

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wild life and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievements;

(k) who is a parent or guardian to provide opportunities for education to his child or, as the case may be, ward between age of six and fourteen years.

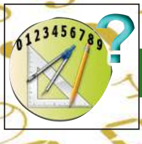
Contents

Certain icons are used in this textbook for convenience

Computer Work

Additional Problems

Project

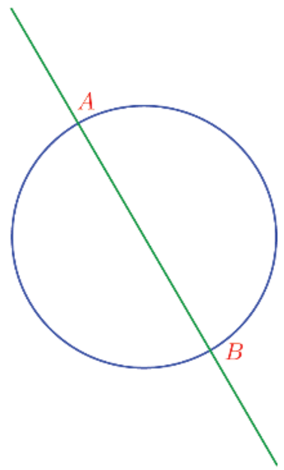


For Discussion

NSQF

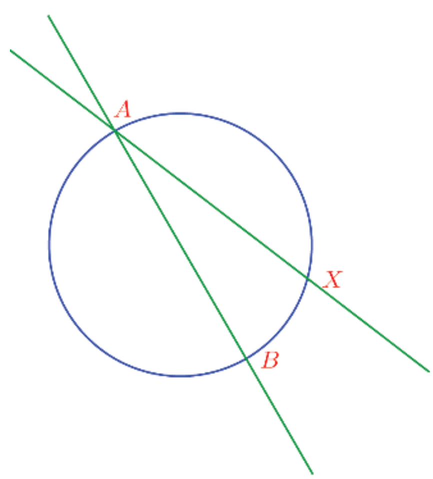
Line and circle

See this picture :

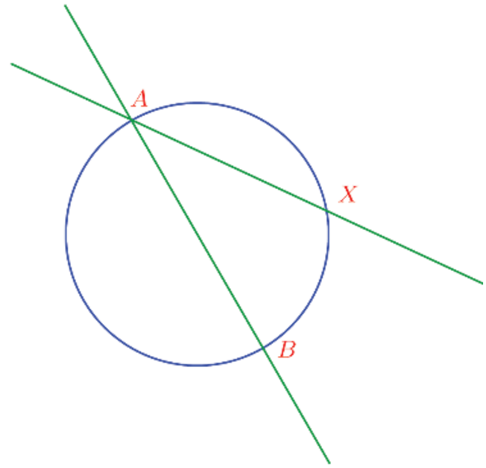


AB is the diameter through the point A on the circle; and it is extended a bit on either side.

This picture shows another chord through A, instead of a diameter, which is also extended.

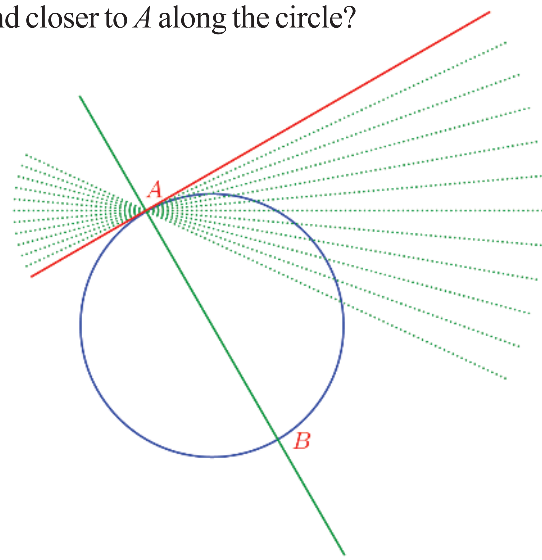


Without altering the position of A, suppose we make X closer to A.



What if we make X closer and closer to A along the circle?

Draw a circle centred at a point O in GeoGebra and mark points A, X on it. Draw lines joining O, A and A, X. What happens to the line AX when X is moved closer to A? When X coincides with A? Join OX. What happens to the angles OAX and AOX as X is moved closer and closer to A?

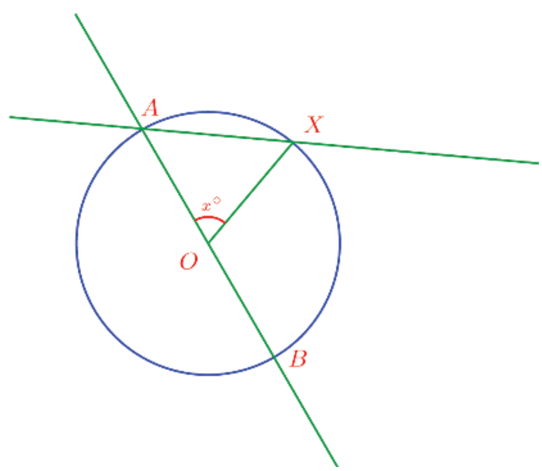


The red line in the picture just touches the circle at A, right?

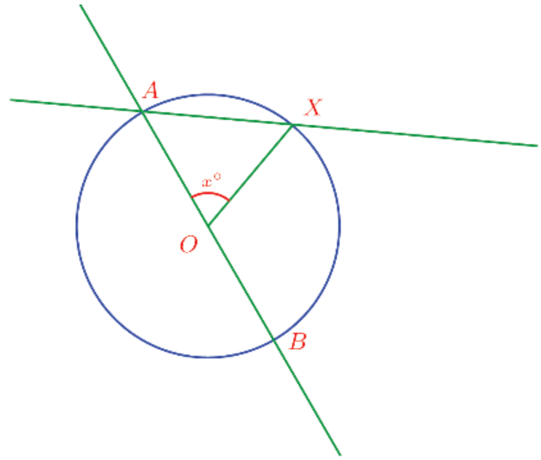
8P1HEF

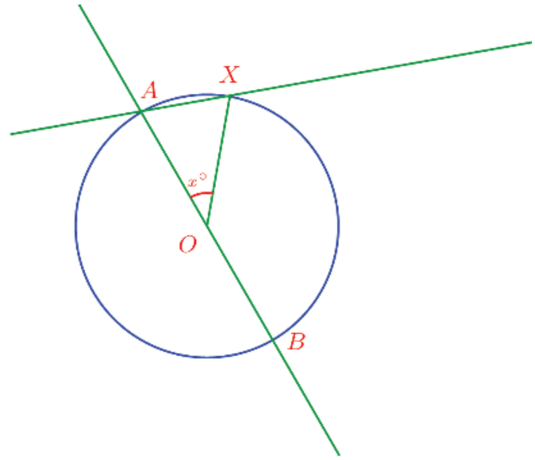
This line is called the *tangent* to the circle at A. Look at the picture again; see any relation between the tangent and the diameter?

To make this clear, let's take the central angle of chord AX as $x°$.



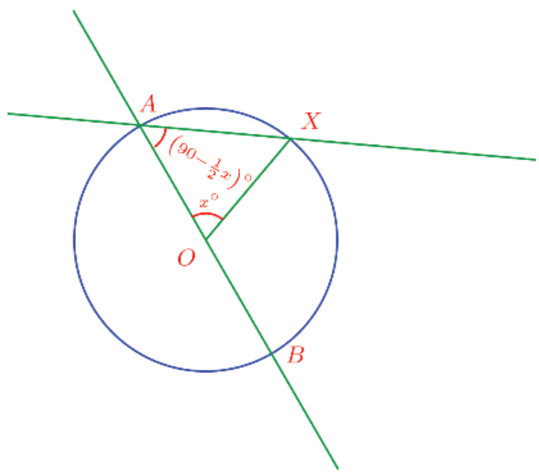
As X gets closer to A, the length of the chord AX and its central angle become smaller; that is, the number x gets closer to zero.





What about the angle between the chord and the diameter? Since ΔAOX is isosceles, this angle is

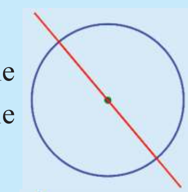
$$\frac{1}{2}(180 - x)^{\circ} = \left(90 - \frac{1}{2}x\right)^{\circ}$$



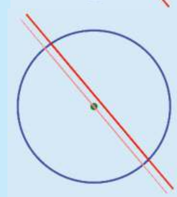
As X gets closer to A, this angle gets closer to 90°. And when the extended chord becomes a tangent, the angle becomes exactly 90°.

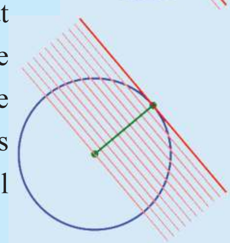
Sliding line

See this picture – a circle and a line through the centre.

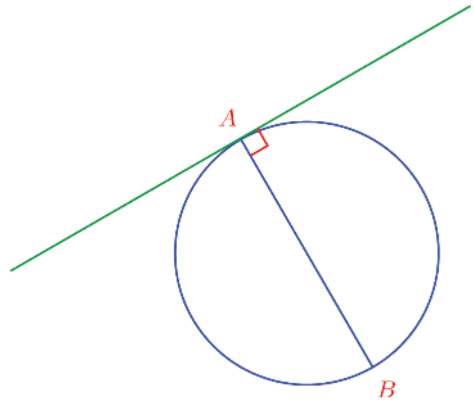


Suppose we slide the line a little higher. And sliding it more and more, we finally get a line just touching the circle at a single point. And the line joining the centre and this point is perpendicular to all these parallel lines.





Draw a circle and a radius in GeoGebra. Choose a point on the radius (point on object) and draw the perpendicular through it. Now change the position of the point. What happens when it is on the circle?

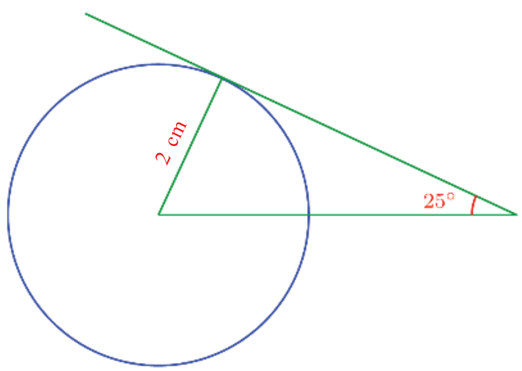


We state this as a general principle:

The tangent at a point on a circle is perpendicular to the diameter through that point.

Let's look at some problems based on this.

In the picture below, the top line is a tangent to the circle:

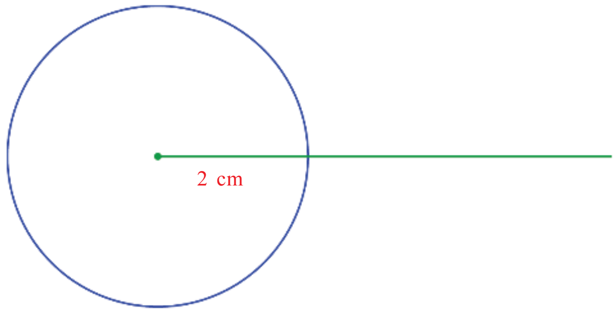


Can you draw this picture in your notebook?

To draw a tangent to a circle in GeoGebra, choose **Tangents.** Click on the circle and a point on the circle. What if we click on a point outside the circle?

Draw the tangent at a point on a circle and enable **Trace On** for it and **Animation** for the point. What do we get?

First draw a circle of radius 2 centimetres and a horizontal line through its centre.



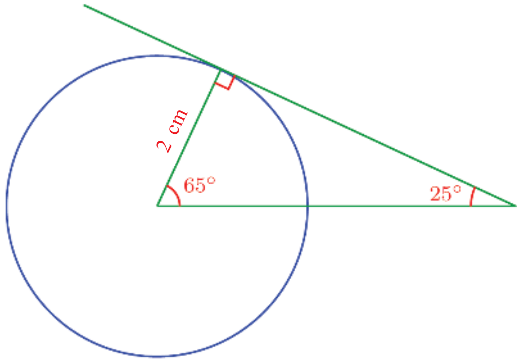
What we need next is a line making an angle of 25° with this line. If we draw a line slanted at 25° from an arbitrary point on this line, it may not be a tangent to the circle.

So, let's think in reverse, at what point on the circle should we draw the tangent? It should meet the first line at a 25° slant.

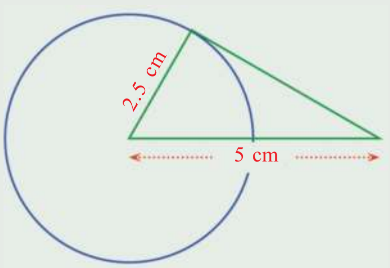
Look at the first picture again; the top angle of the triangle is 90°; and another angle is 25°.

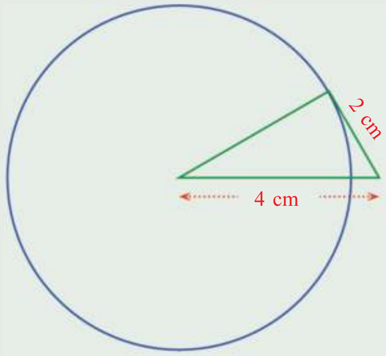
So, the third angle is 65°.

Can't you complete the picture now?



(1) In each of the two pictures below, a triangle is formed by a tangent to a circle, the radius through the point of contact and a line through the centre:

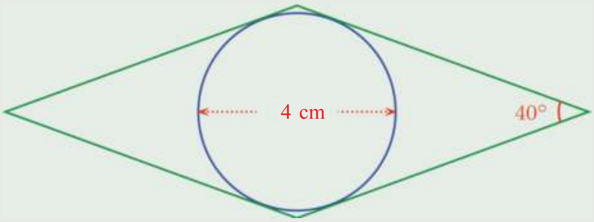




Draw these in your note book.

(2) In the picture, all sides of a rhombus are tangents to a circle.

Draw this picture in your notebook.

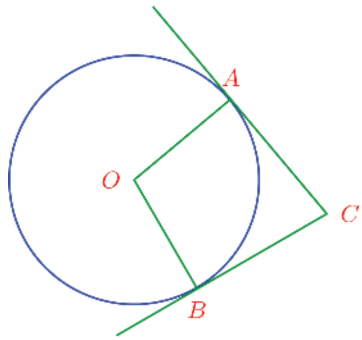


(3) Prove that the tangents drawn to a circle at the two ends of a diameter are parallel.

(4) What sort of a quadrilateral is formed by the tangents at the ends of two perpendicular diameters of a circle?

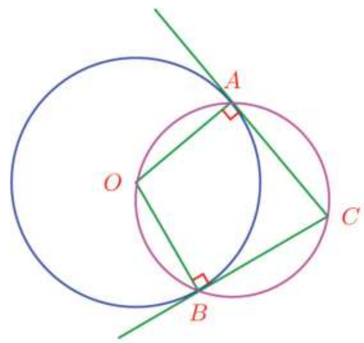
Tangents and angles

See this picture:



The tangents at the points A, B on a circle centred at O meet at C.

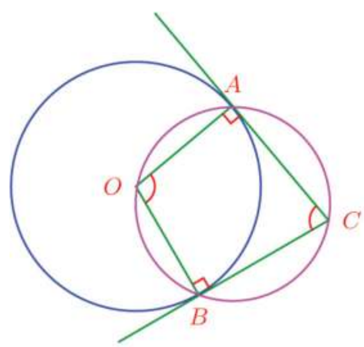
In the quadrilateral $OACB$, the angles at the opposite corners A, B are right; so their sum is 180°. Thus the quadrilateral is cyclic.



That is,

The quadrilateral with vertices at the centre of a circle, two points on it and the point where the tangents at these points meet, is cyclic.

In such a quadrilateral the sum of the other two angles is also 180°.

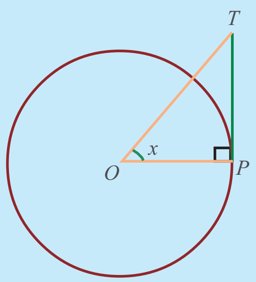


8PASG3

The name

The name tangent is derived from the Latin word *tangere*, meaning 'to touch'.

The full name of the tan measure is also tangent. What is its connection with a line touching a circle?

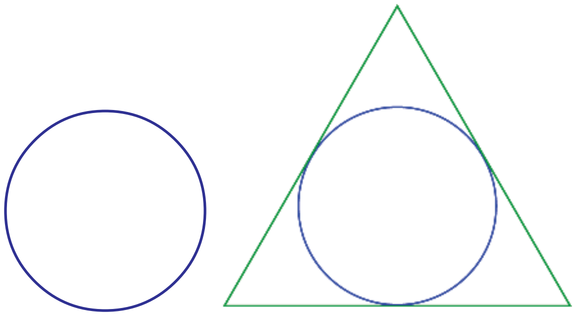


If we take the radius of the circle as the unit of length, then the length of the tangent PT is tan x, right?

This also is a general idea worth noting:

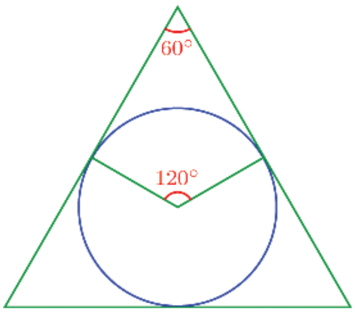
In a circle, the angles between the radii through two points and the angle between the tangents at these points are supplementary.

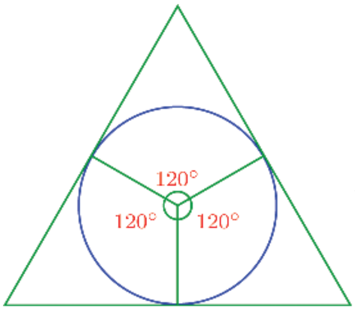
Let's look at some figures which can be drawn using these ideas. First we draw an equilateral triangle, exactly covering a circle.



The sides of the triangle must be tangents to the circle. And since the triangle is to be equilateral, the angle between them must be 60°.

What about the angle between the radii through the points of contact of the tangents?

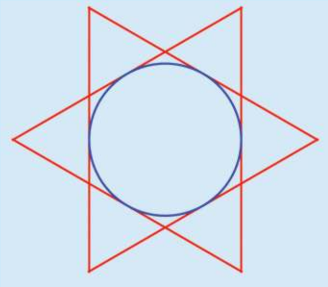




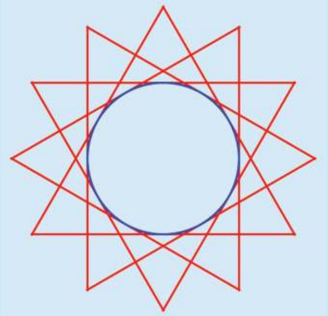
Thus we can see that all the angles between the radii through the points of contact are 120°.

Circle with lines

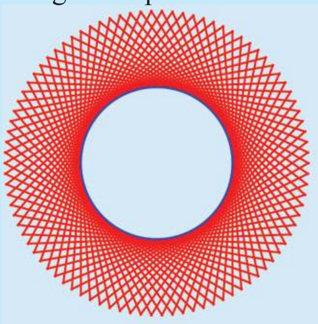
The picture below shows a star made by six tangents to a circle.



How about 12 tangents?



The picture below shows 90 tangents drawn using a computer:

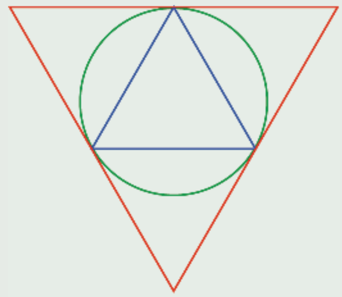


Thus we need only draw three radii of the circle 120° apart and draw the tangents at their ends to get our triangle.

Draw a circle of radius 3 centimetres and draw an equilateral triangle like this.

(1) Draw a circle of radius 2.5 centimetres. Draw a triangle of angles 40°, 60°, 80° with all its sides touching the circle.

(2) In the picture, the small (blue) triangle is equilateral. The sides of the large (red) triangle are tangents to the circumcircle of the small triangle at its vertices.

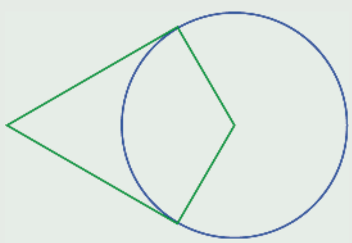


i) Prove that the large triangle is also equilateral and its sides are double those of the small triangle.

ii) Draw this picture, with sides of the smaller triangle 3 centimetres.

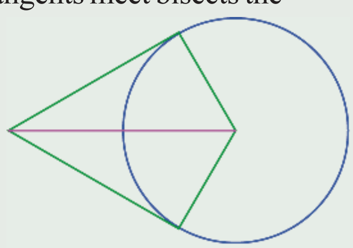
Draw a circle and mark a point on it. Choose **Angle with Given Size** and click on this point and the centre of the circle; in the window opening up, give the angle size as 120°. We get another point on the circle. Click on this new point and the centre of the circle and give 120° as angle size again to get a third point. Draw tangents to the circle at these three points and mark their points of intersection. Draw a triangle joining these points. Now we can hide the tangents. Enable **Trace On** for the sides of the triangle and set **Animation On** for the first point chosen on the circle. What do you see?

(3) The picture shows the tangents at two points on a circle and the radii through the points of contact.

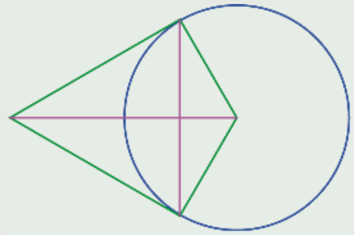


i) Prove that the tangents have the same length.

ii) Prove that the line joining the centre and the point where the tangents meet bisects the angle between the radii.



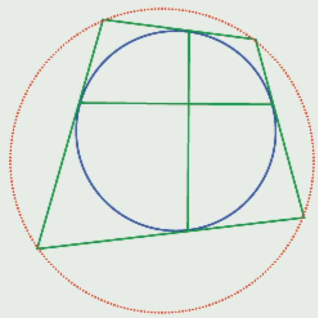
iii) Prove that this line is the perpendicular bisector of the chord joining the points of contact.



(4) Prove that the quadrilateral with sides as the tangents at the ends of a pair of perpendicular chords of a circle is cyclic. (See the Problem (7) in the section, **Chord, angle and arc** of the chapter **Circles**)

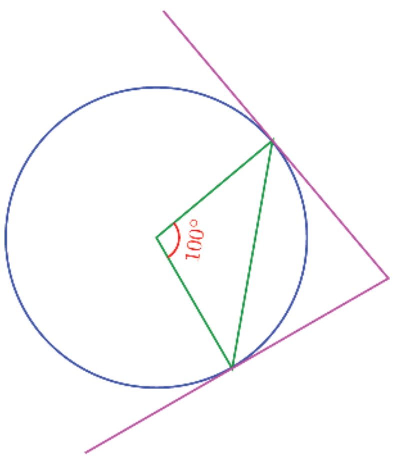
What sort of a quadrilateral do we get if one chord is a diameter? And if both chords are diameters?

8PTJJD

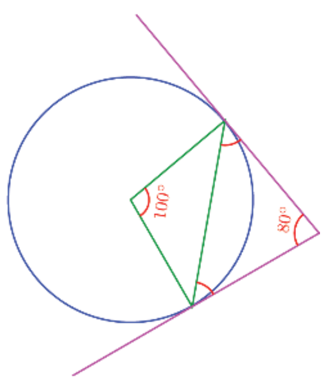


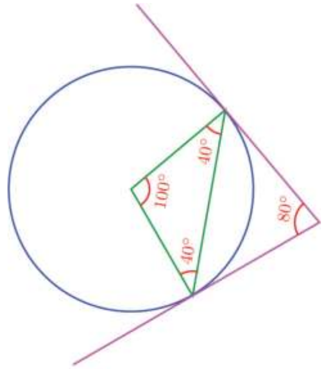
Chord and tangent

The picture shows the tangents at the two ends of a chord of a circle:



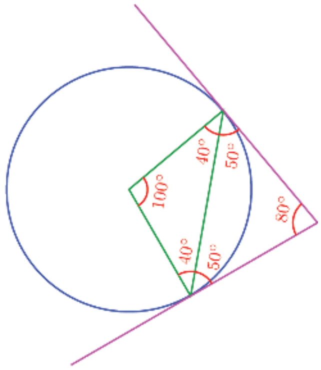
We know that angle between the tangents is 80°. What about the angles between the tangents and the chord?





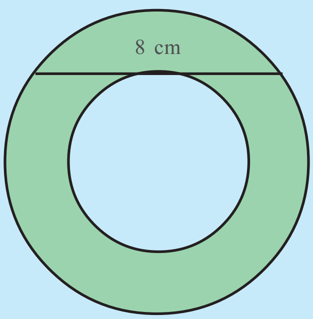
The two sides of the green triangle in the picture are equal and so are the angles opposite them. Since the sum of these angles is $180^\circ - 100^\circ = 80^\circ$, each measures 40°.

The angle between the radius and the tangent is 90°. So the angle between the chord and the tangent is $90^\circ - 40^\circ = 50^\circ$.



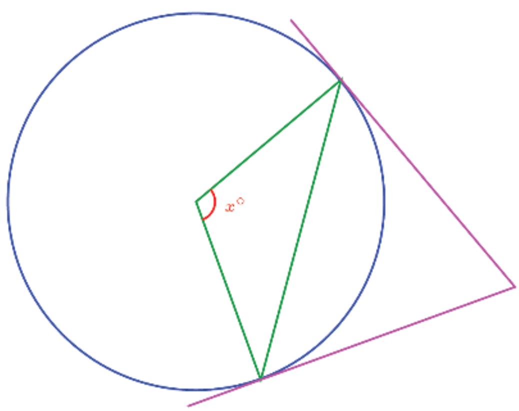
Area problem

What is the area of the green ring?



This is half the central angle of the chord, isn't it?

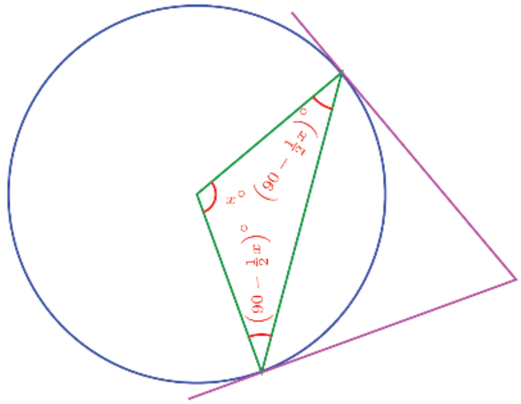
Is it true for any chord? To check this, let's take the central angle as x°.

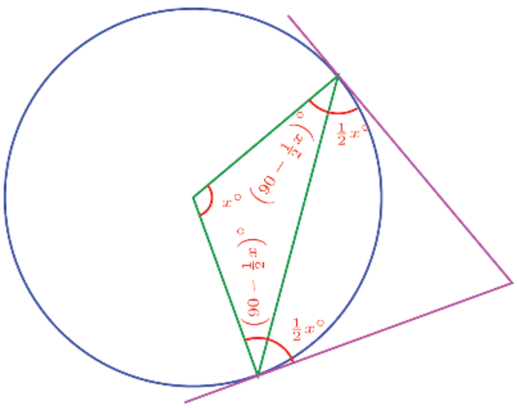


So the other two angles of the green triangle are

$$\frac{1}{2}(180 - x)^\circ = \left(90 - \frac{1}{2}x\right)^\circ$$

Since the angle between the tangent and the radius is 90°, we can see that the angle between the tangent and chord is $\frac{1}{2}x^\circ$.

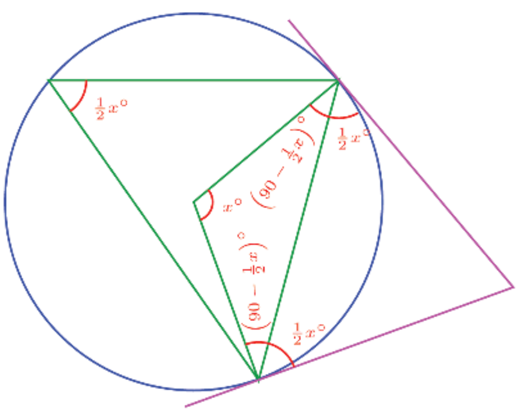




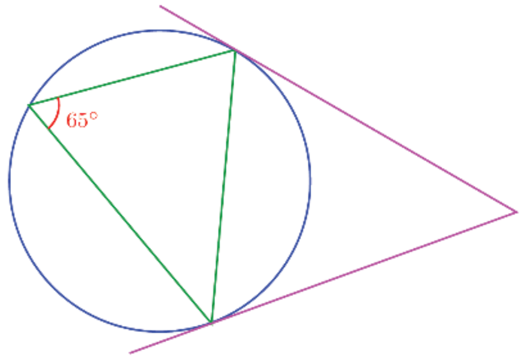
In GeoGebra, draw a circle and a chord. Draw the tangents at the ends of this chord. Mark the central angle of the chord and the angles between the chord and the tangents. What is the relation between these angles? Draw several chords and see.

In a circle, the angle between a chord and tangent at either end is half the central angle of the chord.

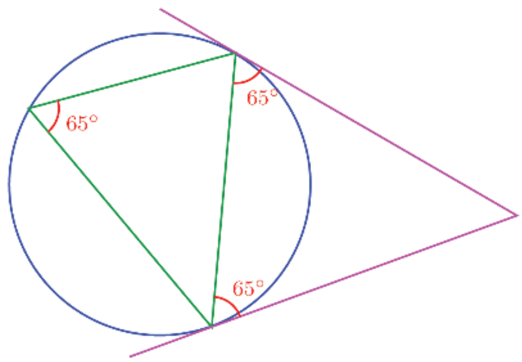
The angle made by the chord on the larger part of the circle is also half the central angle, isn't it? (The chapter, **Circles**)



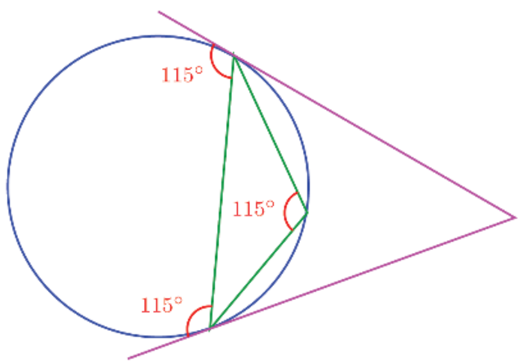
So in this picture, what is the angle which the tangents make with the chord?



The angles on the right are 65°.



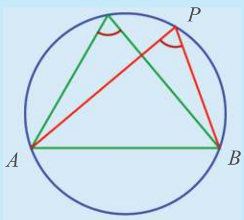
The angles which the tangents make on the left of the chord are 180° – 65° = 115°. This is the angle which the chord makes on the smaller part of the circle, isn't it?



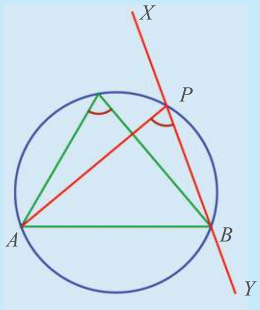
So the relation between the angles which the chord makes with the tangents at its ends and the angles which it makes on the circle can be shown like this:

Unchanging angle

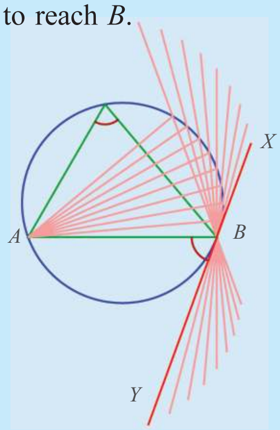
We have seen that the angles on the same part of a circle are equal.



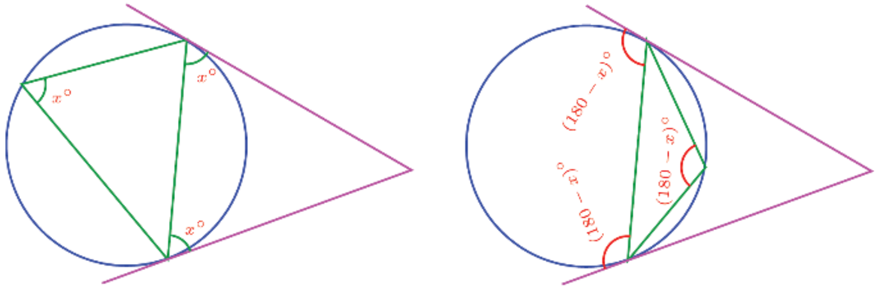
Let's extend PB a bit.



Now suppose P moves along the circle to reach B.



The line XY becomes the tangent at B. And the angle does not change.

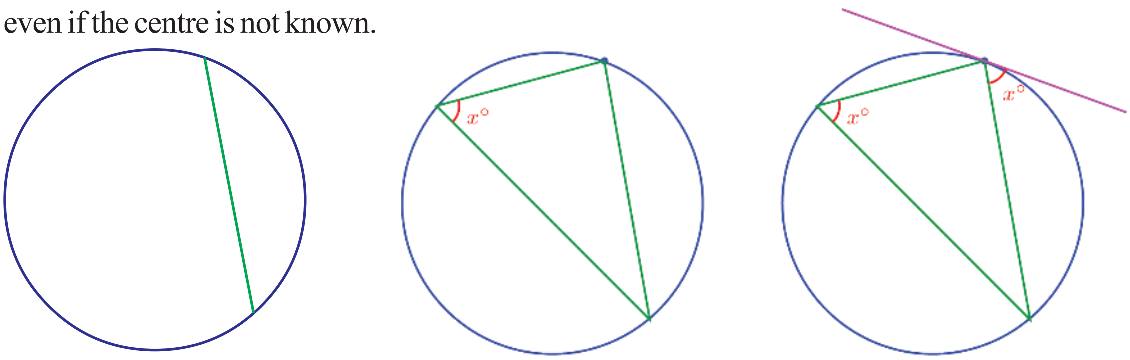


8Q3FL1

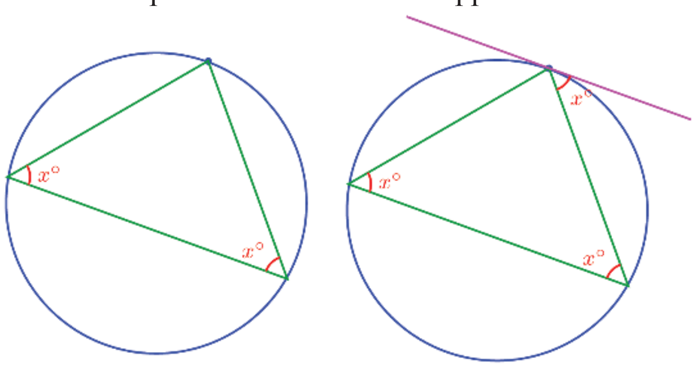
And we can write it like this:

In a circle, the angle which a chord makes with the tangent at one end on any side is equal to the angle which it makes on the part of the circle on the other side.

We draw a tangent to a circle at a point by drawing the perpendicular to the diameter through this point, right? The above idea can be used to draw tangents, even if the centre is not known.



We need only draw a chord through this point and draw the angle which it makes on one part of the circle on the opposite side of the chord:



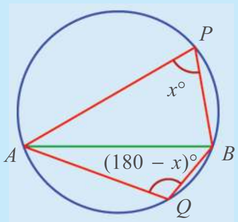
Draw tangents to a circle at points A, B and mark the angles between the chord and tangents. Mark a point C on the circle and draw ∠ACB. What is the relation between these angles? Change the position of A, B, C and see.

If the triangle drawn is isosceles, then we need draw only the line through the point, parallel to the bottom side.

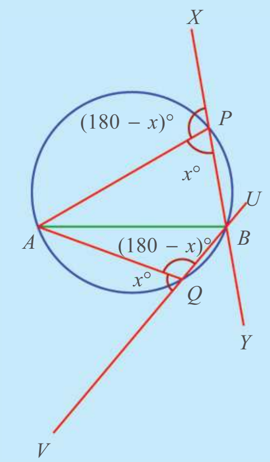
So, to draw the tangent to a circle at a specified point, we first draw an arc of the circle centred at this point and join the points where it cuts the first circle:

Flip – flop

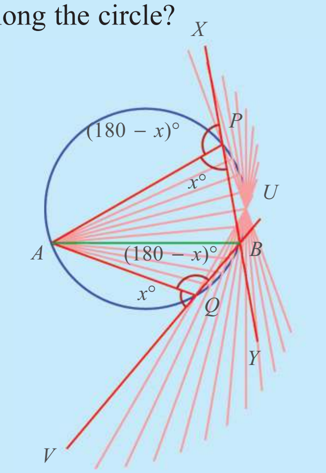
Angles on the two parts of a circle are supplementary:



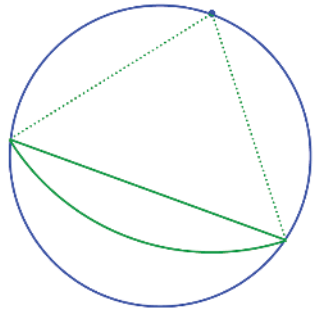
Let's extend the lines as before.



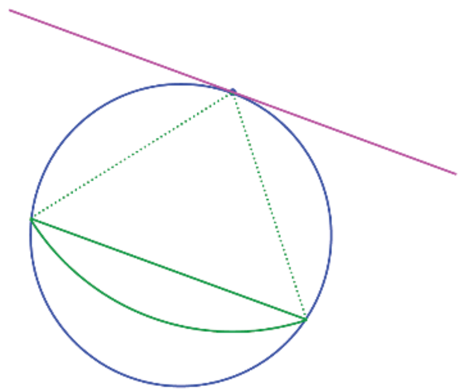
What happens as P moves towards Q along the circle?



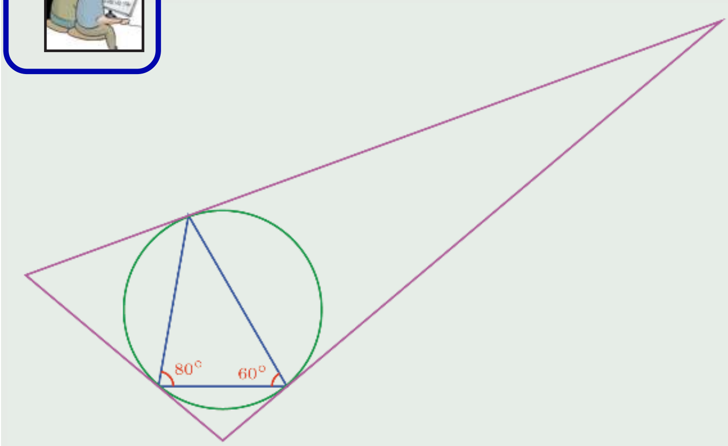
Throughout the motion, $x°$ is below AP and $(180-x)°$ is above AP, isn't it?



Now we need only draw a line parallel to this line through the point at which we want the tangent:



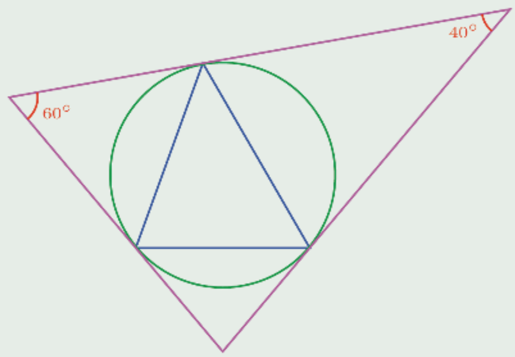
(1) In the picture, the sides of the large triangle are tangents to the circumcircle of the small triangle, through its vertices.



Calculate the angles of the large triangle.

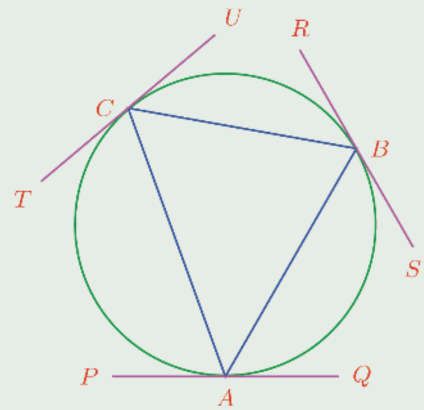
(2) In the picture, the sides of the large triangle are tangents of the circumcircle of the smaller triangle, through its vertices.

Calculate the angles of the smaller triangle.



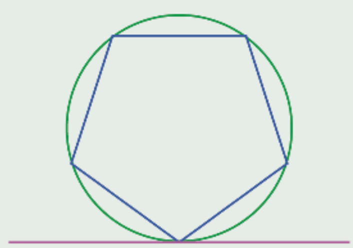
(3) In the picture, PQ, RS, TU are tangents to the circumcircle of ΔABC.

Sort out the equal angles in the picture.



(4) In the picture, the tangent to the circumcircle of a regular pentagon through a vertex is shown.

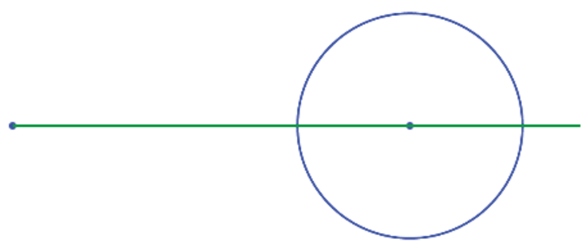
Calculate the angle which the tangent makes with the two sides of the pentagon through the point of contact.



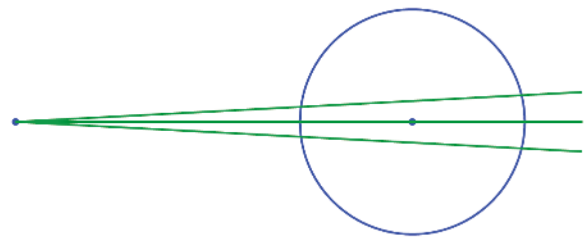
A tangent from outside

See this picture:

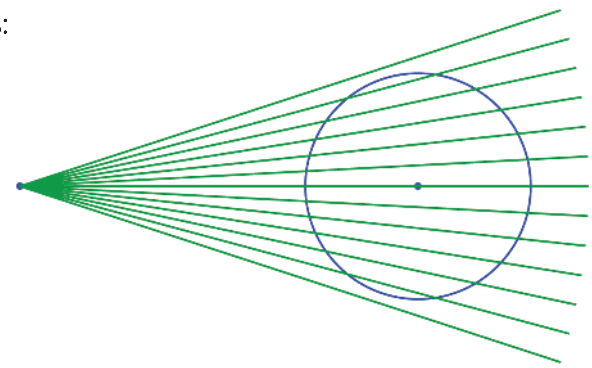
A point outside a circle is joined to the centre and extended. It cuts the circle at two points; and these points are the ends of a diameter.



Suppose we join the same point outside the circle to a point a little above or below the centre?

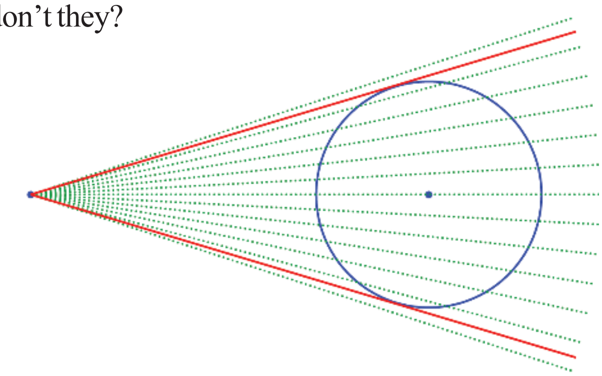


The points where the line cuts the circle get a little closer. Let's continue this:



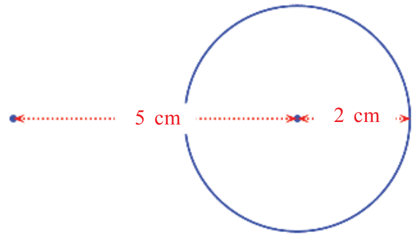
The lines which cut the circle at closer and closer points leave the circle entirely after a stage.

But at some stage before this, two of these lines above and below just touch the circle, don't they?



From a point outside a circle, two tangents can be drawn.

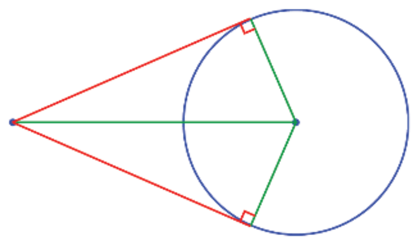
But we haven't discussed how we can actually draw such a pair of tangents. See this picture:



A point is marked 5 centimetres away from the centre of a circle of radius 2 centimetres.

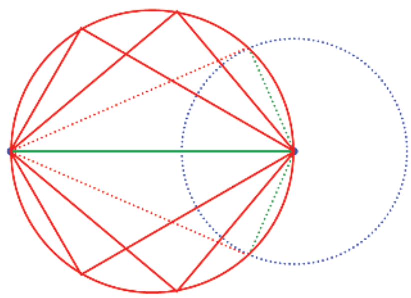
How do we draw the pair of tangents to the circle from this point?

Perhaps it would be clear if we imagine how the picture would be after they are drawn:



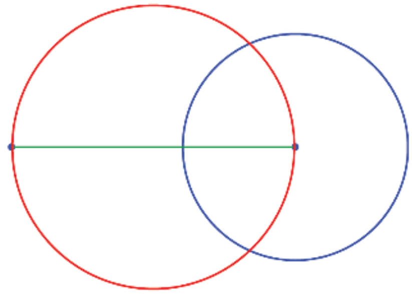
We need two pairs of mutually perpendicular lines from the centre of the circle and the point outside.

We have seen in the lesson **Circles**, that all such pairs of mutually perpendicular lines meet on the circle with the line joining these points as diameter:



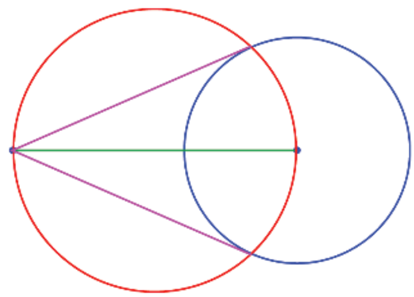
In those pairs we want, one line should be a radius of our original circle; that is, the lines should meet on this circle. For that, we need only draw through the point of intersection of the old and new circles.

Now can't we actually draw the tangents? First draw the circle on the line joining the point outside with the centre as diameter:

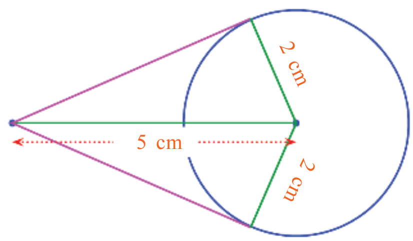


Draw a circle centred at a point O in GeoGebra and mark points A, B on it. Draw the tangents at these points and mark their point of intersection as C. Draw the quadrilateral OACB. Is it cyclic? We can check by drawing the circle through O, A, B using **Circle Through Three Points**. Move A, B and see what happens to C when they get closer and farther apart. What happens when they are the ends of a diameter?

Joining the points of intersection of these circles to the point outside, we get the tangents from it:



In our problem, the radius of the original circle is 2 centimetres and the distance from the centre to the point outside is 5 centimetres.



So, we can calculate the lengths of the tangents using Pythagoras Theorem:

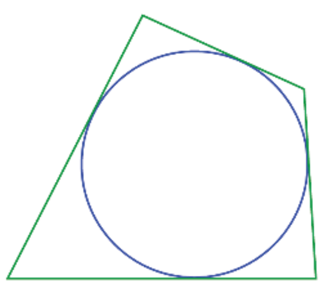
$$\sqrt{5^2 - 2^2} = \sqrt{21} \text{ centimetres}$$

We have already seen that if tangents are drawn from two points on a circle, then their lengths from the point of contact to the point of intersection are equal. We can now state it like this:

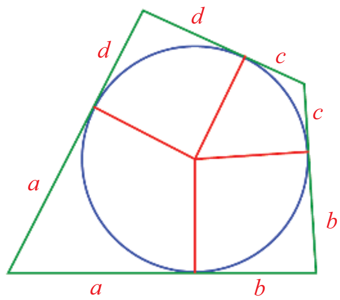
The tangents to a circle from a point are of the same length.

Let's look at a problem based on this. The picture shows the quadrilateral formed by the tangents at four points on a circle.

Let's join the centre to these points.



Taking the lengths of the tangents from the corners as a, b, c, d, we can mark these lengths as below:



So, the sum of the lengths of the bottom and top sides of the quadrilateral is $(a + b) + (c + d)$.

What about the sum of the left and right sides? $(a + d) + (b + c)$

Both sums are $a + b + c + d$. Thus we have the following:

In a quadrilateral formed by the tangents at four points on a circle, the sum of the opposite sides are equal.

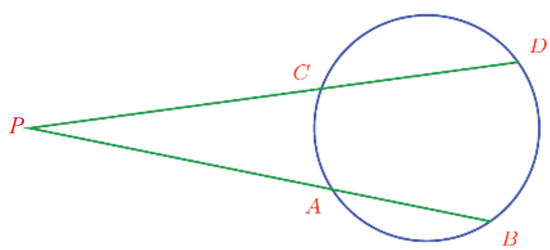
Draw a circle in GeoGebra and mark four points on it. Draw tangents at these points and mark their points of intersection. Draw the quadrilateral with these as vertices. Then we can hide the tangents. Mark the lengths of the sides of the quadrilateral and note the relation between them, as we change the positions of the points on the circle.

Recall the fact we have seen earlier: in a quadrilateral formed by joining four points on a circle, the sum of the opposite angles are equal.

If the sum of the opposite sides of a quadrilateral are equal, can we draw a circle with the four sides as tangents?

We have seen that among the lines drawn to a circle from a point, those which touch the circle at a single point are equal. We have also seen in the lesson, **Circles**, that for all lines cutting the circle at two points, the product of the whole line and the part outside the circle are equal.

Remember this picture and its equation?

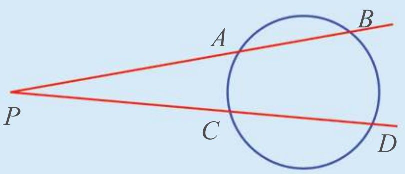


$$PA \times PB = PC \times PD$$

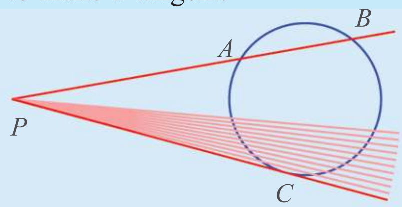
Now suppose, we draw a line which touches a circle and another line intersecting the circle.

Same relation

See this picture:



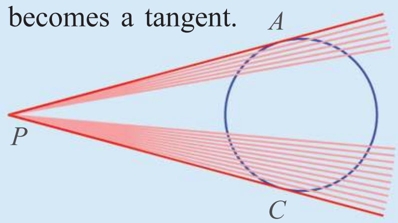
We have $PA \times PB = PC \times PD$. Suppose the lower line is rotated to make a tangent.



Then PD is the same as PC and the relation above becomes

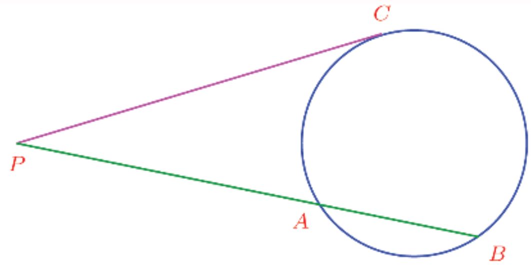
$$PA \times PB = PC^2$$

Suppose the upper line also becomes a tangent.

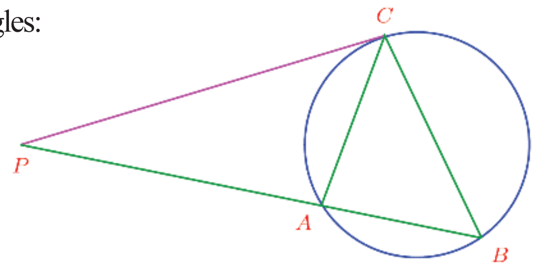


The relation becomes $PA^2 = PC^2$ or $PA = PC$.

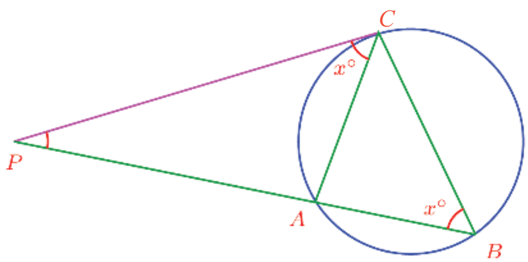
We have already seen that the length of the tangents from a point to a circle are equal.



To find out the relation between these, join AC, BC to make triangles:



The chord AC makes angle PCA at C; it is equal to the angle ABC, which AC makes on the other side of the circle, isn't it?



That is, the angle at C in ΔABC is equal to the angle at B in ΔPBC. And in both triangles the angle at P is the same.

Thus these triangles have the same angles and so pairs of sides opposite equal angles have the same ratio.

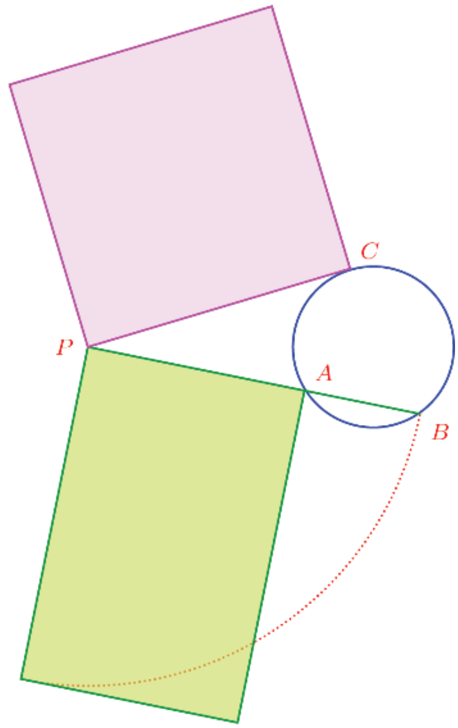
In ΔPAC, the side opposite the angle of $x°$ is PA and in ΔPBC, the side opposite the angle of $x°$ is PC. In ΔPAC, the longest side is PC and in ΔPBC, the longest side is PB. So,

$$\frac{PA}{PC} = \frac{PC}{PB}$$

We can write this as,

$$PA \times PB = PC^2$$

8QCBHN



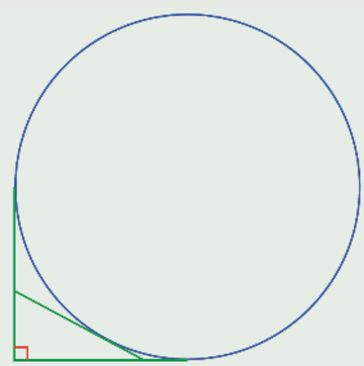
The product of an intersecting line and the part of it outside the circle is equal to the square of the tangent.

As in the case of intersecting chords, this can be stated in terms of areas.

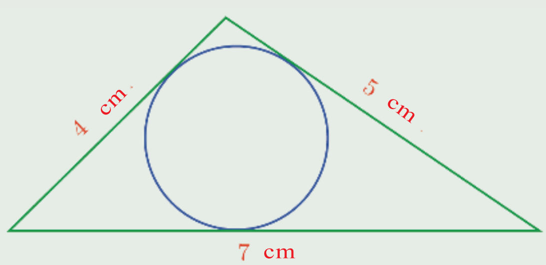
The rectangle with the intersecting line and its part outside the circle as sides and the square with sides equal to the tangent have the same area.

(1) In the picture, a triangle is formed by two mutually perpendicular tangents to a circle and a third tangent.

Prove that the perimeter of the triangle is equal to the diameter of the circle.

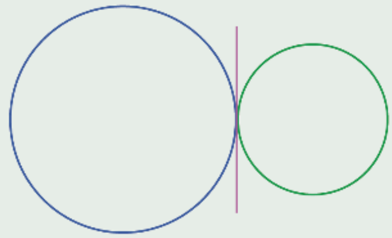


(2) The picture shows a triangle formed by three tangents to a circle.

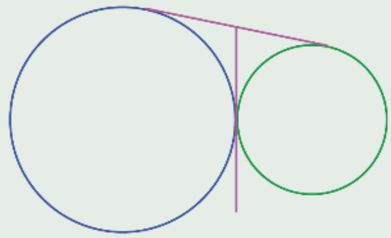


Calculate the length of each tangent from the corner of the triangle to the point of contact.

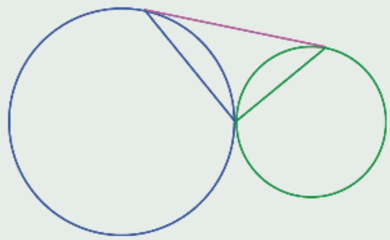
(3) In the picture, two circles touch at a point and the common tangent at this point is drawn.



i) Prove that this tangent bisects another common tangent of these circles.

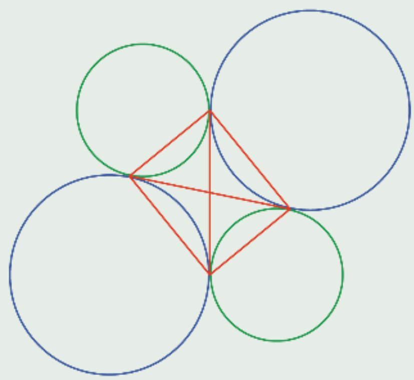


ii) Prove that the points of contact of these two tangents form the vertices of a right triangle.

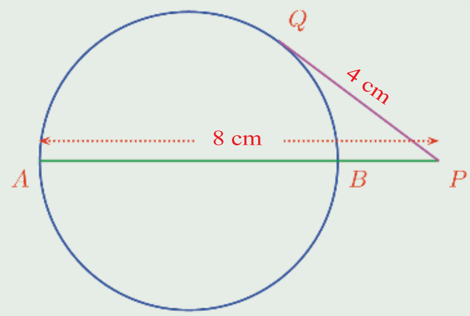


iii) Draw the picture on the right in your notebook, using convenient lengths.

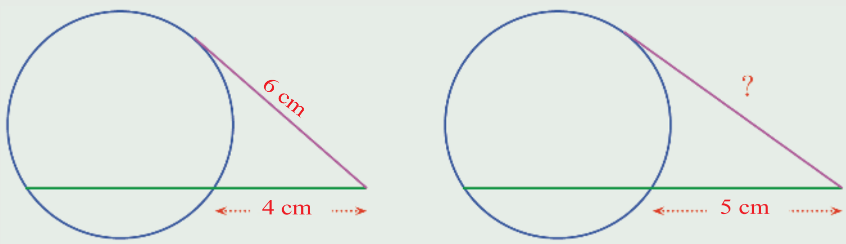
What is special about the quadrilateral formed by joining the points of contact of the circles?



(4) In the picture below, AB is a diameter and P is a point on AB extended. A tangent from P touches the circle at Q. What is the radius of the circle?



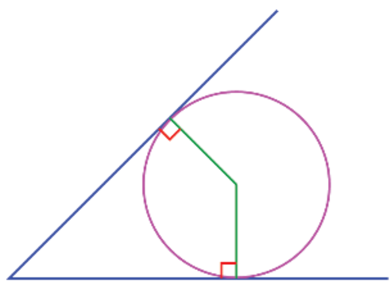
(5) In the first picture below, the line joining two points on a circle is extended by 4 centimetres and a tangent is drawn from this point. Its length is 6 centimetres, as shown:



The second picture shows the same line extended by 1 centimetre more and a tangent drawn from this point. What is the length of this tangent?

Circle touching a line

We have seen that from a point, two lines touching a circle can be drawn and also how we can draw these lines.

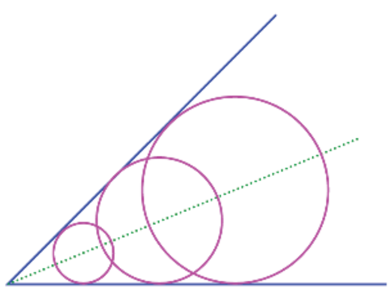


Now let's ask in reverse: can we draw a circle touching two lines meeting at a point? See this picture.

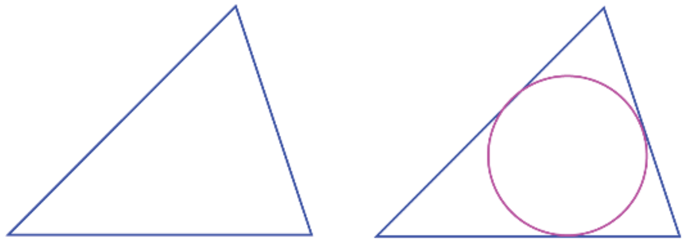
The radii are perpendicular to these lines. In other words, the centre of the circle must be at the same distance from these lines. So it must be on the bisector of this angle. (See Problem (4) of the section, **Triangle division** in the Chapter **Area** in the class 9 textbook)

The centre of a circle touching two lines meeting at a point is on the bisector of the angle formed by the lines.

We can take any point on the angle bisector as centre to draw a circle touching the lines.

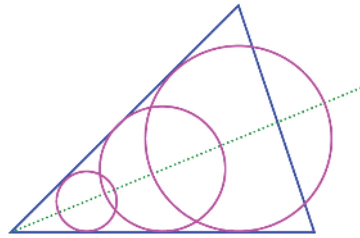


So, the next question is whether we can draw a circle touching all three sides of a triangle.

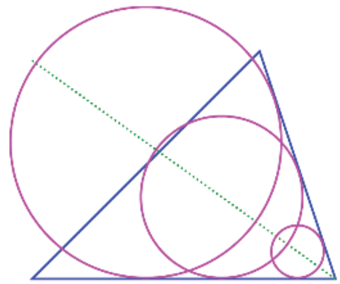


Draw an angle and its bisector in GeoGebra. Mark a point on this bisector and draw the perpendicular from this point to one of the sides of the angle. Mark their point of intersection. Draw the circle with centre at the point on the bisector and passing through the point on the side. Doesn't it touch both sides? Move the centre along the bisector and see.

We can take any point on the bisector of the angle made by the bottom and left sides to draw a circle touching these two sides.



Taking any point on the bisector of the angle made by the bottom and right sides as centre, we can draw a circle touching these two sides.

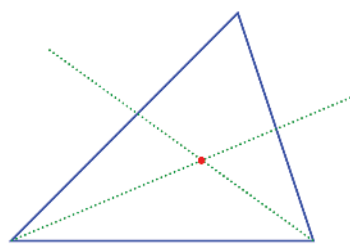


So, what if we take the point of both these bisectors, that is their point of intersection?

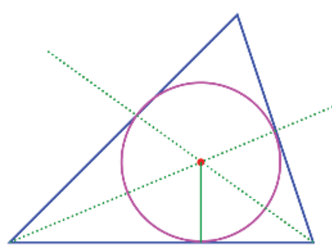
The lengths of the perpendiculars from this point to all three sides are equal, right?

What about the circle of radius this length, centred at this point?

This circle is called the *incircle* of the triangle.



We note another thing here. Since the centre of the incircle is at the same distance from the left and right sides, it is also on the bisector of the angle joining these sides.

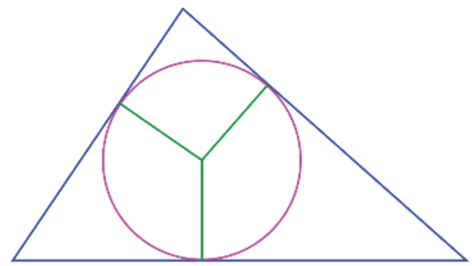


Draw a triangle and its incircle in GeoGebra.

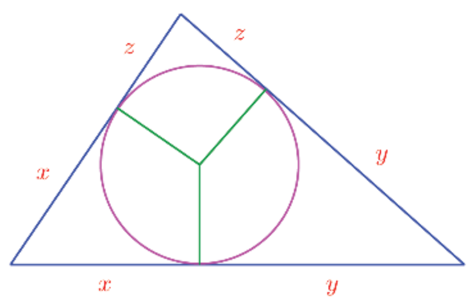
The bisectors of all three angles of a triangle meet at a point.

There are some relations between the points where the incircle touches the triangle and the sides of the triangle.

To see it, we draw the incircle of a triangle and join its centre to the points of contact with the sides.



The sides of the triangle are formed by the tangents to the incircle from the vertices. And the lengths of the tangents from each corner are equal. So taking the lengths of the tangents as x, y, z, we can mark them as below:



The sum of all these lengths is the perimeter of the triangle. That is, the perimeter of the triangle is $2(x+y+z)$. In other words, $x+y+z$ is half the perimeter of the triangle. Taking it as s,

$$x + y + z = s$$

Next if we take the lengths of the sides of the triangle as a, b, c, the picture gives

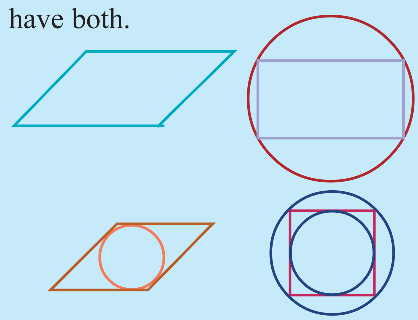
$$x + y = a$$

$$y + z = b$$

$$z + x = c$$

Circumcircle and incircle

Every triangle has a circumcircle and an incircle. But in the case of quadrilaterals, some have neither of these, some have only one and some have both.



Now to get x, we need only subtract $y + z$ from $x + y + z$:

$$x = (x + y + z) - (y + z) = s - b$$

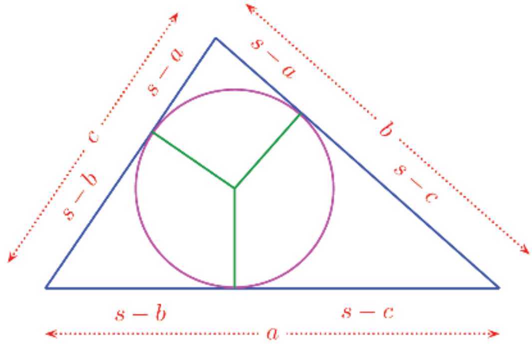
Similarly, we see that

$$y = (x + y + z) - (z + x) = s - c$$

and

$$z = (x + y + z) - (x + y) = s - a$$

So the lengths of tangents are like this:



Area

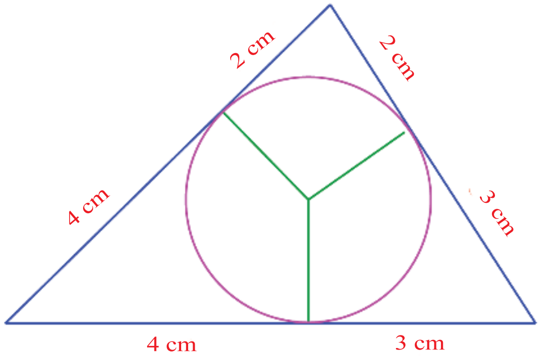
We have seen how the incircle of a triangle divides the three sides. Using this and the similarity of certain triangles got by drawing some more lines in the triangle, we can compute the area of the triangle. This was discovered by Heron, a Greek mathematician of the first century AD. If we take the length of the sides of the triangle as a, b, c and put $s = \frac{1}{2}(a + b + c)$, the area is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

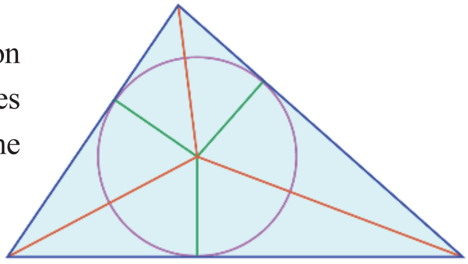
(https://en.wikipedia.org/wiki/Heron's_formula)

For example, consider a triangle of sides 5 centimetres, 6 centimetres and 7 centimetres. Half the perimeter is 9 centimetres.

So the points of contact with the incircle divides the sides like this: $9 - 5 = 4$, $9 - 6 = 3$, $9 - 7 = 2$



The radius of the incircle has a relation with the area of the triangle. The lines joining the centre of the incircle to the vertices divide the triangle into three.



One side of each of these small triangles is a side of the original large triangle and the height from it is equal to the radius of the incircle. So, if we take the sides of the triangle as a, b, c and the radius of the incircle as r, then the areas of these small triangles are $\frac{1}{2}ar$; $\frac{1}{2}br$; $\frac{1}{2}cr$.

Their sum is the whole area of the large triangle. Taking it as A, we have

$$A = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}(a+b+c)\,r = sr$$

This equation can written as

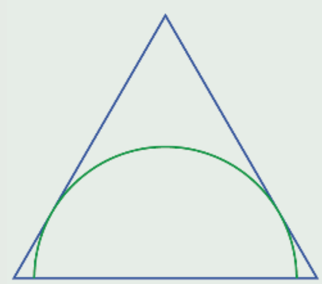
$$r = \frac{A}{s}$$

The radius of the incircle of a triangle is its area divided by half the perimeter.

(1) Draw a triangle of sides 4 centimetres, 5 centimetres, 6 centimetres and draw its incircle. Calculate its radius.

(2) Draw a rhombus of sides 5 centimetres and one angle 50° and draw its incircle.

(3) Draw an equilateral triangle and a semi-circle touching its two sides, as in the picture.



(4) What is the radius of the incircle of a right triangle having perpendicular sides of length 5 centimetres and 12 centimetres?

(5) Prove that if the hypotenuse of a right triangle is h and the radius of its incircle is r, then its area is $r(h+r)$.

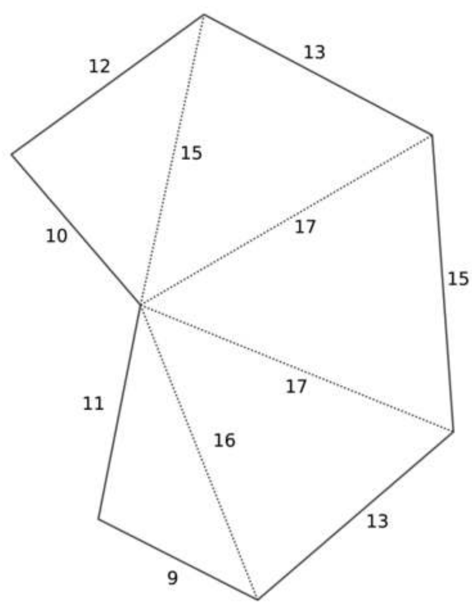
(6) Prove that the radius of the incircle of an equilateral triangle is half the radius of its circumcircle.

Area

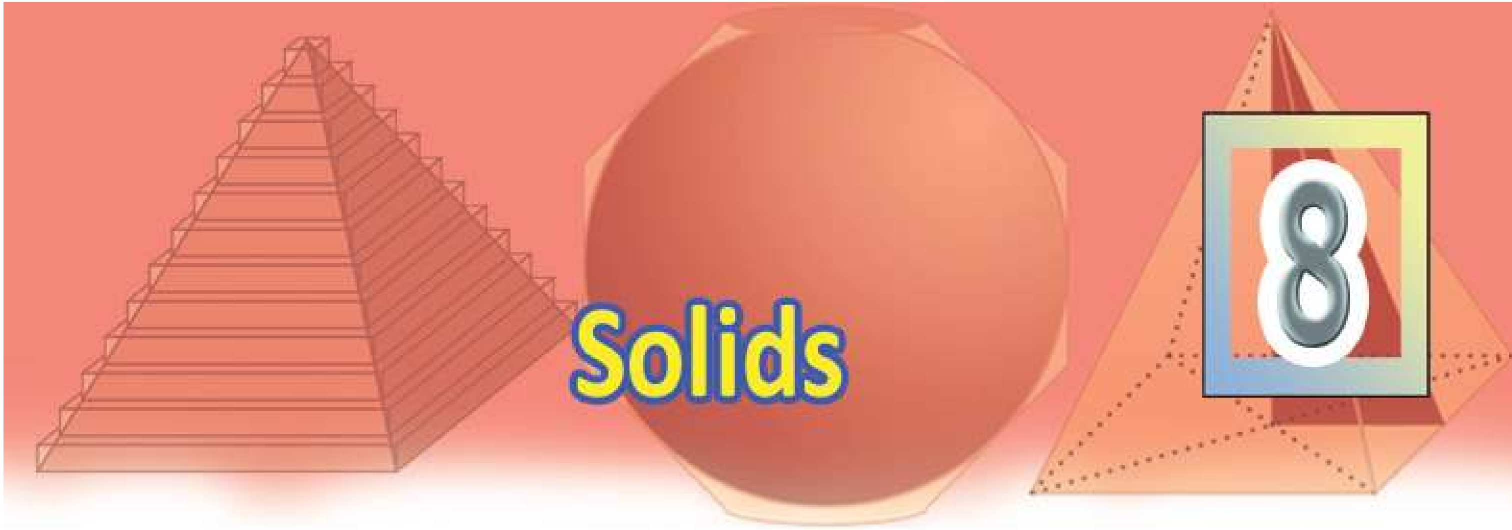
We have noted that the area of a triangle with length of sides a, b, c is $\sqrt{s(s-a)(s-b)(s-c)}$. As an example, let's compute the area of a triangle of sides 12, 35 and 43 centimetres. Taking $a = 12$, $b = 35$, $c = 43$ centimetres we get $s = \frac{12+35+43}{2} = 45$, $s - a = 33$, $s - b = 10$, $s - c = 2$. Using a calculator, we can compute the area.

$$\text{Area} = \sqrt{45 \times 33 \times 10 \times 2} \approx 172.33 \text{ square centimetres}$$

This can be used to find the area of any region bounded by straight lines. The picture below shows a ground divided into triangles:

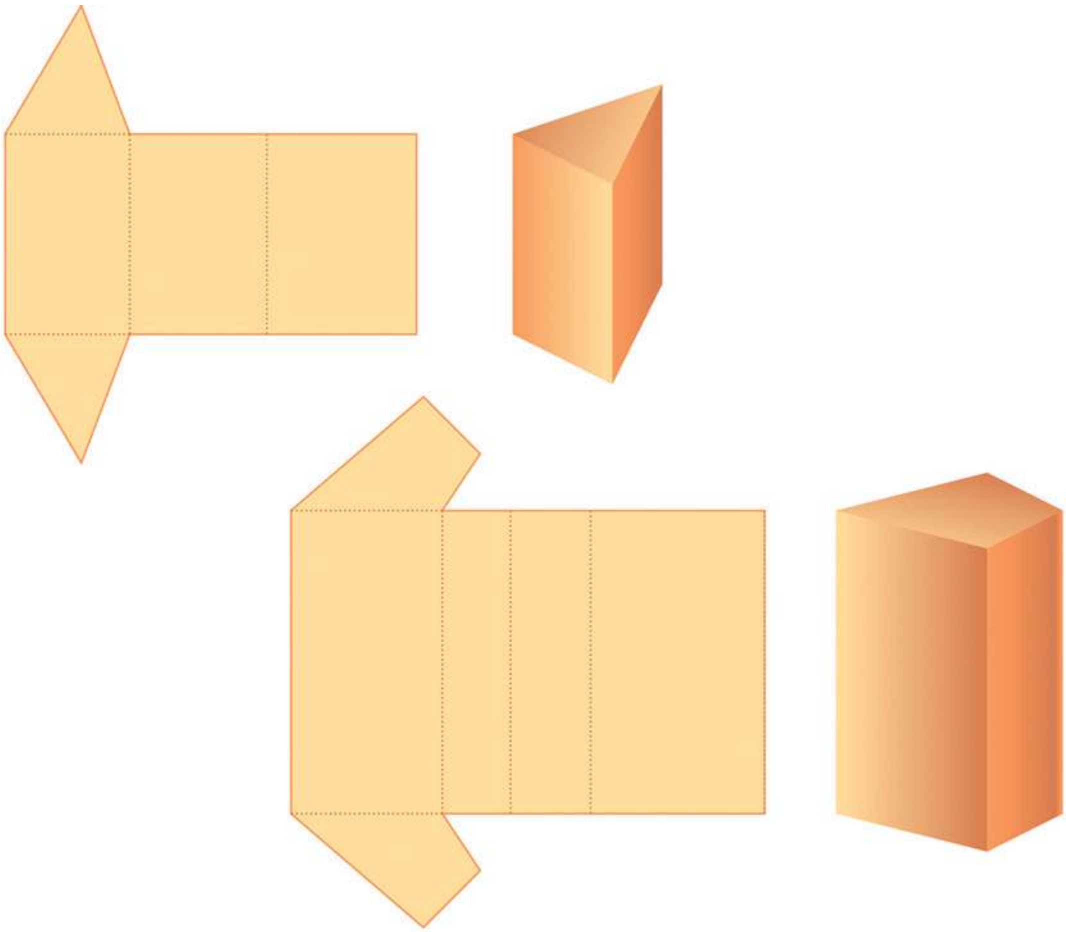


All the sides of the triangles are shown (in metres). So we can compute the area of each by the above formula and add them all to get the area of the whole ground.



Pyramids

We can make prisms by cutting thick paper in various ways and pasting the edges.



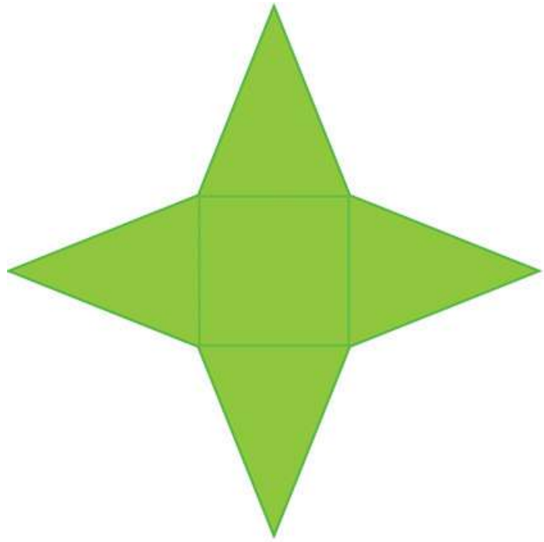
And we have learnt much about them.

Let’s make another kind of solid:

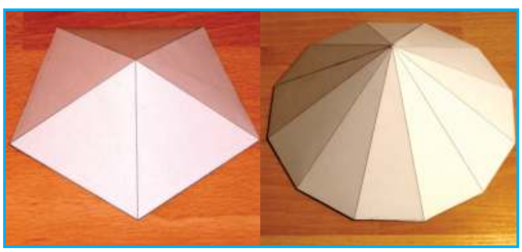
Pyramids in GeoGebra

We have seen in Class 9 how prisms can be drawn in GeoGebra. Now let's see how pyramids can be drawn. Open 3D Graphics and make the necessary preparations (See the section, **Solids in GeoGebra** of the lesson, **Prisms** in the Class 9 textbook) Draw a square in **Graphics.** Choose **Extrude to Pyramid or Cone** in **3D Graphics** and click on the square. In the window opening up, type the height of the pyramid. (we can also make a slider and give its name as the height).

First, cut out a figure like this in paper.



A square in the middle and four triangles around it; all four of them are isosceles triangles and they are equal. Now fold and paste as shown below.

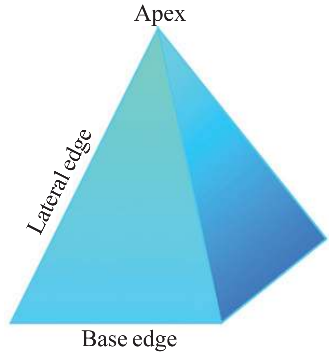


What shape is this? Can't be called a prism; prisms have two equal bases and rectangles on the sides. In the shape we have made now, we have a square at the bottom, a point on top and triangles all around.

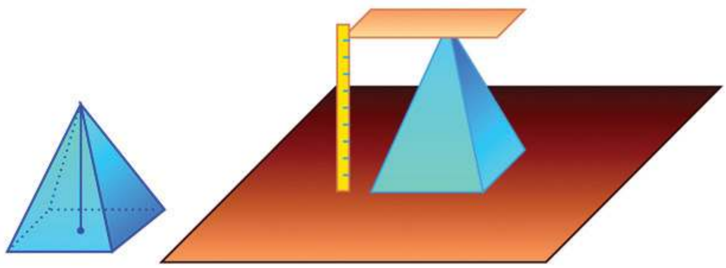
Instead of square, the base can be some other rectangle, a triangle or some other polygon. Try! (It is better looking when the base is a regular polygon.)

Such a solid is generally called a *pyramid*.

The sides of the polygon forming the base of a pyramid are called *base edges* and the other sides of the triangles are called *lateral edges*. The topmost point of a pyramid is called its *apex*.

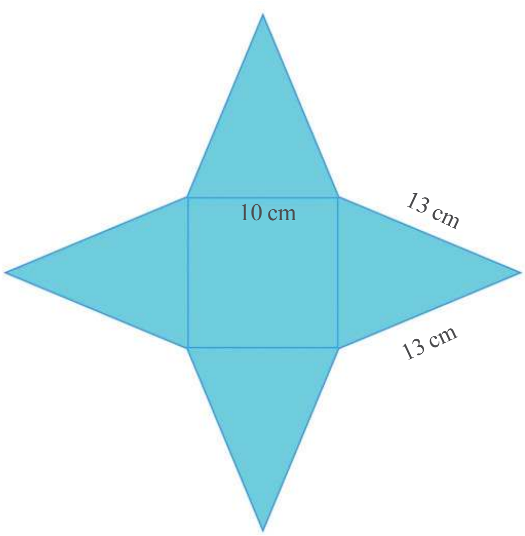


The height of a prism is the distance between its bases, isnt it? The height of a pyramid is the perpendicular distance from the apex to the base.



Area

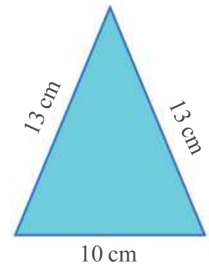
What is the surface area of a square pyramid of base edges 10 centimetres and lateral edges 13 centimetres? The surface area is the area of paper needed to make it. How will it look, if we cut this pyramid open and lay it flat?



Let's see how GeoGebra helps us to see the 'cut and spread' shape of a pyramid. Make a pyramid in 3D graphics as described earlier. Choose **Net** and click on the pyramid. We get the shape of the paper used to make it (It is called the net of the solid). We also get a slider in **Graphics**. By moving the slider, we can see how the pyramid is made from the net. We can also hide the original pyramid by clicking against the pyramid in the **Algebra** window.

The area of the square is easily seen to be 100 square centimetres. What about the triangles?

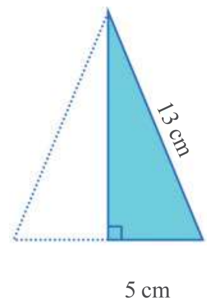
The area of the triangle is half the product of its base and height, isn't it?



For that, we need the height of the triangle. Since the triangle is isosceles, this altitude bisects the base.

So using Pythagoras Theorem, the height of the triangle is

$$\sqrt{13^2 - 5^2} = 12 \text{ centimetres}$$

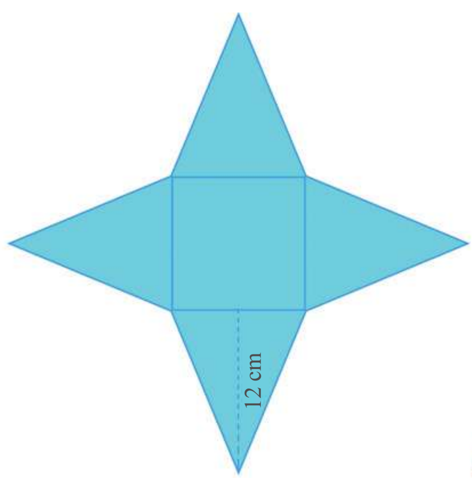


Thus the area of the triangle is $5 \times 12 = 60$ square centimetres.

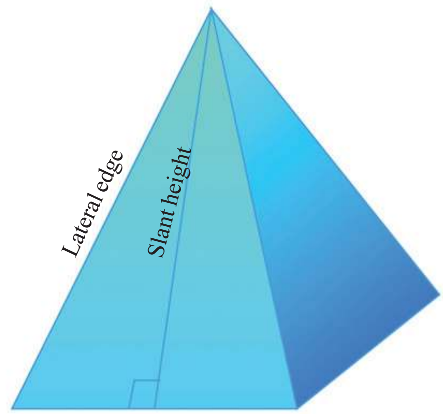
What will be the height of the triangle, when the paper is turned into a pyramid?

Height and slant height

Draw a pyramid in GeoGebra. Click **Midpoint or Centre** to mark the mid points of a base edge and the midpoint of a diagonal of the base. Use **Segment** to mark the height and slant height of the pyramid. Use **Polygon** to make the right triangle with height, slant height and half the base edge as sides. Using **Net**, the pyramid can be cut and spread. We can also hide the pyramid.



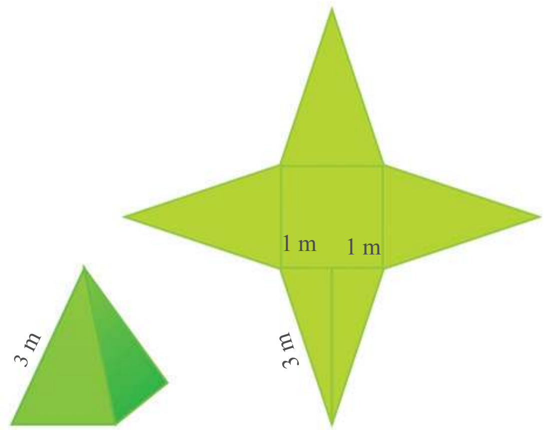
This length is called the *slant height* of the pyramid.



Half the base edge

We have seen the relation between the base edge, lateral edge and slant height of a pyramid in the problem we did just now. As shown in the picture on the right, there is a right triangle on each side of the pyramid - its perpendicular sides are the slant height and half the base edge, the hypotenuse is a lateral edge.

Now do this problem: what is the surface area of a square pyramid with base edges 2 metres and lateral edges 3 metres?



The base area is 4 square metres. To compute the areas of lateral faces, we need the slant height. In the right triangle mentioned above, one side is half the base edge, that is, 1 metre and the hypotenuse is the lateral edge of 3 metres. So, the slant height is

$$\sqrt{3^2 - 1^2} = 2\sqrt{2} \text{ metres}$$

Using this, the area of each triangular face is

$$\frac{1}{2} \times 2 \times 2\sqrt{2} = 2\sqrt{2} \text{ square metres.}$$

So, the surface area of the pyramid is,

$$4 + \left(4 \times 2\sqrt{2}\right) = 4 + 8\sqrt{2} \text{ square metres.}$$

If not satisfied with this, a calculator can be used, (or an approximate value of $\sqrt{2}$ recalled) to compute this as 15.31 square metres.

(1) A square of side 5 centimetres, and four isosceles triangles of base 5 centimetres and height 8 centimetres, are to be put together to make a square pyramid. How many square centimetres of paper is needed?

(2) A toy is in the shape of a square pyramid of base edge 16 centimetres and slant height 10 centimetres. What is the total cost of painting 500 such toys, at 80 rupees per square metre?

(3) The lateral faces of a square pyramid are equilateral triangles and the length of a base edge is 30 centimetres. What is its surface area?

(4) The perimeter of the base of square pyramid is 40 centimetres and the total length of all its edges is 92 centimetres. Calculate its surface area.

(5) Can we make a square pyramid with the lateral surface area equal to the base area?

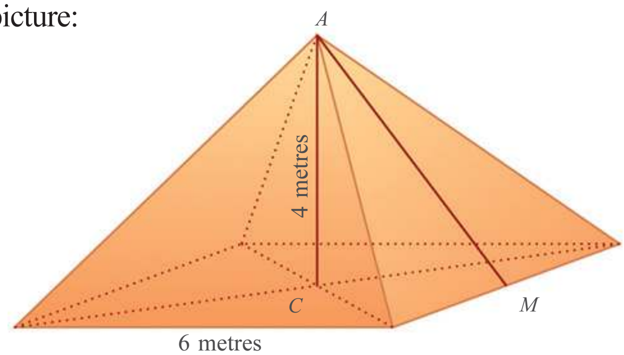
Height and slant height

The height of a pyramid is often an important measure. See this problem:

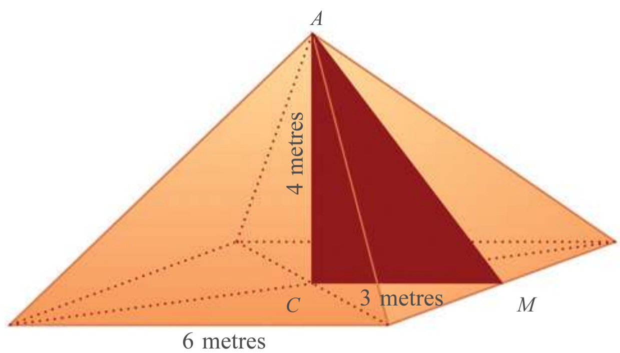
> A tent is to be made in the shape of a square pyramid of base edges 6 metres and height 4 metres. How many square metres of canvas is needed to make it?

To calculate the area of the triangular faces of the tent, we need the slant height. How do we compute it using the given specifications?

See this picture:



The slant height we need is AM. Joining CM, we get a right triangle with AM as hypotenuse. What is the length of CM in it?



From the picture, $AM = \sqrt{3^2 + 4^2} = 5$ metres.

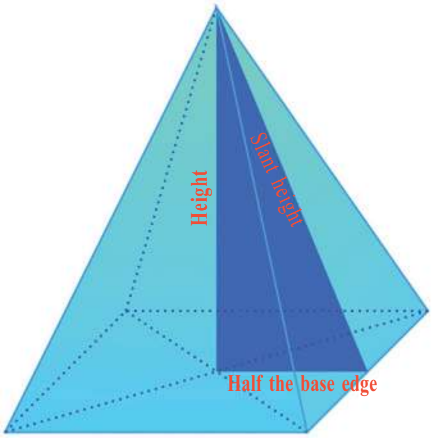
Pyramids of Egypt

The very word pyramid brings to our mind the great pyramids of Egypt. 138 such pyramids are found from various parts of Egypt. Many of them were built around 2000 BC.

So, to make the tent, four isosceles triangles of base 6 metres and height 5 metres are needed. Their total area is $4 \times \frac{1}{2} \times 6 \times 5 = 60$ square metres.

So this much canvas is needed to make the tent.

In this problem, we have found something which is true in the case of all square pyramids. Within every square pyramid, we can imagine a right triangle with perpendicular sides as the height of the pyramid and half the base edge and hypotenuse as the slant height.

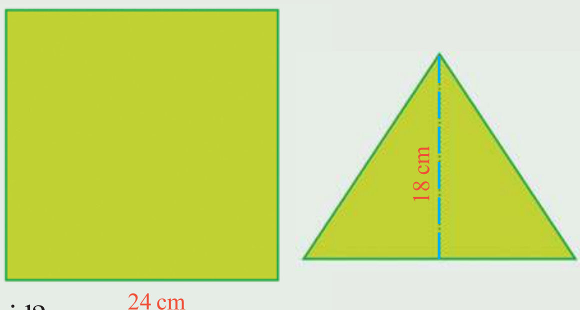


Great Pyramid

The largest pyramid in Egypt is the Great Pyramid of Giza.

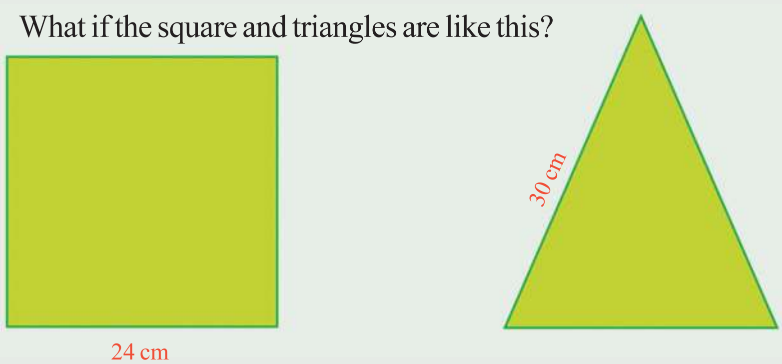
Its base is a square of almost half a lakh square metres and its height is about 140 metres. It is estimated that about 20 years would have been needed to complete it. These royal tombs built with huge blocks of stones stacked with precision to end in a point are living symbols of human labour, engineering skill and mathematical knowledge.

(1) Using a square and four triangles with dimensions as specified in the picture, a pyramid is made.



What is the height of this pyramid?

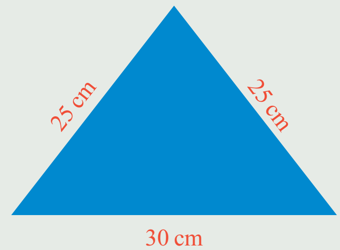
What if the square and triangles are like this?



(2) A square pyramid of base edge 10 centimetres and height 12 centimetres is to be made of paper. What should be the dimensions of the triangles?

(3) Prove that in any square pyramid, the squares of the height, slant height and lateral edge are in arithmetic sequence.

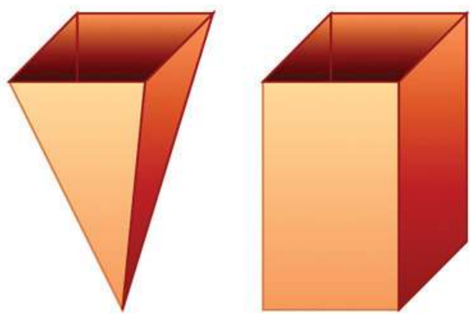
(4) A square pyramid is to be made with the triangle shown here as a lateral face. What would be its height? What if the base edge is 40 centimetres instead of 30 centimetres?



Can we make a square pyramid with any four equal isosceles triangles?

Volume of a pyramid

We have seen that the volume of any prism is equal to the product of the base area and the height. What about the volume of a pyramid?



Let's take the case of a square pyramid. Make a hollow square pyramid with thick paper and also a square prism of the same base and height.

Pyramid Volume

Draw a square pyramid and a square prism of the same base and with the same height in GeoGebra. To distinguish between them, change the colour of the pyramid and make Opacity 100. (**Object properties → Colour**). Find their volumes using **Volume.** What is the relation between them? Change the base and height and see.

Fill the pyramid with sand and transfer it to the prism. Measure the height of the sand in the prism and see what fraction of the height of the prism it is. A third, isn't it? So to fill the prism, how many times should we fill the pyramid?

Thus we see that the volume of the prism is three times the volume of the pyramid. (A mathematical explanation of this is given at the end of this lesson).

We have seen in Class 9 that the volume of a prism is equal to the product of the base area and the height.

So what can we say about the volume of a square pyramid?

The volume of a square pyramid is equal to a third of the product of the base area and the height.

For example, the volume of a square pyramid of base edge 10 centimetres and height 8 centimetres is $\frac{1}{3} \times 10^2 \times 8 = 266\frac{2}{3}$ cubic centimetres.

A metal cube of edge 15 centimetres is melted and recast into a square pyramid of base edge 25 centimetres. What is its height?

The volume of the cube is 15^3 cubic centimetres. The volume of the square pyramid is also this. And the volume of a pyramid is a third of the product of the base area and height. Since the base area of our pyramid is 25^2 square centimetres, a third of the height is $\frac{15^3}{25^2}$ and so the height is,

$$3 \times \frac{15^3}{25^2} = 16.2 \text{ centimetres.}$$

(1) What is the volume of a square pyramid of base edge 10 centimetres and slant height 15 centimetres?

(2) Two square pyramids have the same volume. The base edge of one is half that of the other. How many times the height of the second pyramid is the height of the first?

(3) The base edges of two square pyramids are in the ratio 1 : 2 and their heights in the ratio 1 : 3. The volume of the first is 180 cubic centimetres. What is the volume of the second?

(4) All edges of a square pyramid are 18 centimetres. What is its volume?

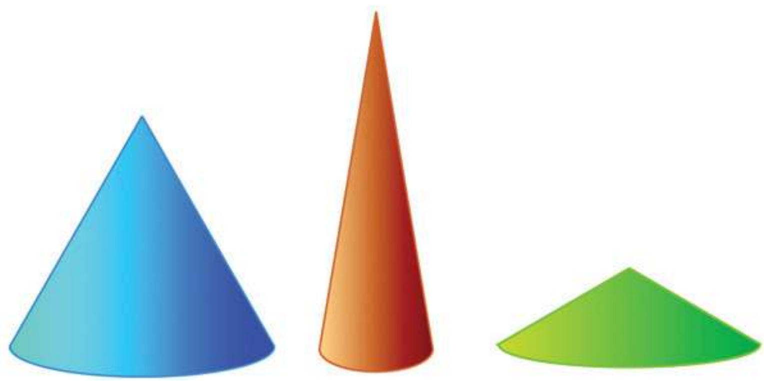
(5) The slant height of a square pyramid is 25 centimetres and its surface area is 896 square centimetres. What is its volume?

(6) All edges of a square pyramid are of the same length and its height is 12 centimetres. What is its volume?

(7) What is the surface area of a square pyramid of base perimeter 64 centimetres and volume 1280 cubic centimetres?

Cone

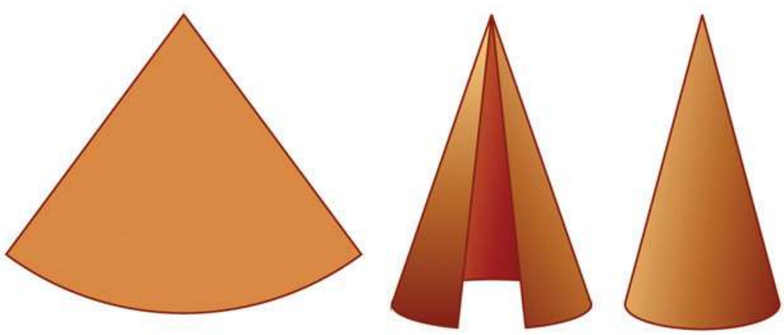
Cylinders are prism-like solids with circular bases. Similarly, we have pyramid-like solids with circular bases:



They are called *cones*.

We can make a cylinder by rolling up a rectangle. Likewise, we can make a cone by rolling up a sector of a circle.

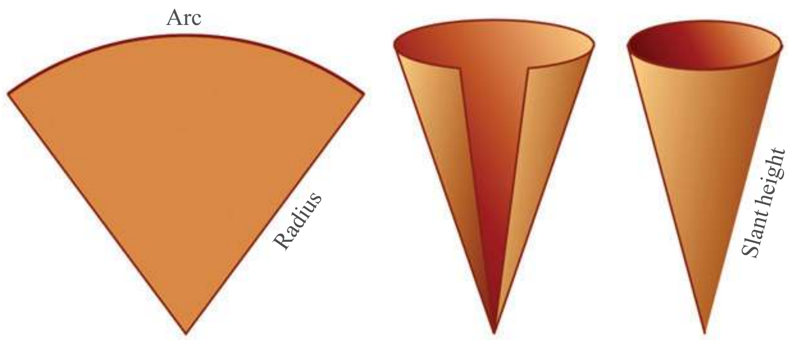
What is the relation between the dimensions of the sector we start with and the cone we end up with?



Cone

We can draw cones in GeoGebra, just as we drew pyramids. Draw a circle in **Graphics** and in **3D Graphics**, use **Extrude to Pyramid or Cone**. The base radius and height can be changed using sliders.

The radius of the sector becomes the slant height of the cone. The arc length of the sector becomes the circumference of the base of the cone.



We often specify the size of a sector in terms of the central angle. See this problem:

> From a circle of radius 12 centimetres, a sector of central angle 45° is cut out and made into a cone. What are the slant height and base radius of this cone?

The slant height of the cone is the radius of the circle itself: 12 centimetres. What about its base radius?

45° is $\frac{1}{8}$ of 360°. And the arc length of a sector is proportional to the central angle. So this arc length is $\frac{1}{8}$ of the circumference of the full circle.

This arc becomes the base circle of the cone. Thus the circumference of the base circle of the cone is $\frac{1}{8}$ of the circumference of the larger circle from which the sector was cut out. Since radii of circles are proportional to their circumferences, the radius of the smaller circle is $\frac{1}{8}$ of the radius of the large circle. Thus the radius of the base of the cone is $\frac{1}{8} \times 12 = 1.5$ centimetres.

How about a question in the reverse direction?

> How do we make a cone of base radius 5 centimetres and slant height 15 centimetres?

To make a cone, we need a sector. Since the slant height is to be 15 centimetres, the sector must be cut out from a circle of radius 15 centimetres.

What should be its central angle?

The radius of the small circle forming the base of the cone is $\frac{5}{15} = \frac{1}{3}$ of the radius of the large circle from which the sector is to be cut out. (How do we get this?) So, the circumference of the small circle is also $\frac{1}{3}$ of the circumference of the large circle.

The circumference of the small circle is the arc length of the sector. Thus the arc of the sector is $\frac{1}{3}$ of the circle from which it is cut out. So its central angle must be $360 \times \frac{1}{3} = 120°$.

(1) What are the radius of the base and slant height of a cone made by rolling up a sector of central angle 60° cut out from a circle of radius 10 centimetres?

(2) What is the central angle of the sector to be used to make a cone of base radius 10 centimetres and slant height 25 centimetres?

(3) What is the ratio of the base-radius and slant height of a cone made by rolling up a semicircle?

Curved surface area

As in the case of a cylinder, a cone also has a curved surface - the part which rises up at a slant. The area of this curved surface is the area of the sector used to make the cone. (For a cylinder also, the area of the curved surface is the area of the rectangle rolled up to make it, isn't it?)

For a cone drawn in GeoGebra, the curved surface area can be seen under **Surface** in the **Algebra** window.

See this problem:

To make a conical hat of base radius 8 centimetres and slant height 30 centimetres, how much square centimetres of paper do we need?

What we need here is the area of the sector we roll up to make this hat. Since the slant height is to be 30 centimetres, we must cut out the sector from a circle of this radius. Also the radius of the small circle forming the base of the cone must be 8 centimetres, that is $\frac{8}{30} = \frac{4}{15}$ of the radius of the large circle from which the sector is cut out. So the circumference of the small circle is also the same fraction of the circumference of the large circle. The arc length of the sector is the circumference of the small circle. Thus the sector to be cut out is $\frac{4}{15}$ of the full circle. So, its area is that fraction of the area of the circle; that is,

$$\pi \times 30^2 \times \frac{4}{15} = \pi \times 2 \times 30 \times 4 = 240\pi$$

Thus we need 240π square centimetres of paper to make the hat (It can be computed as approximately 754 square centimetres).

Curved surface

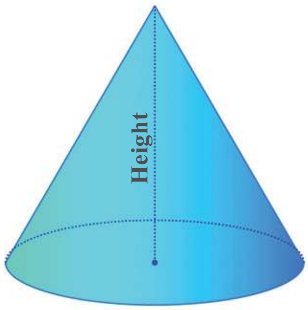
The area of the curved surface of a cone is the area of the sector used to make it. If we take the base radius of the cone as r and its slant height as l, then the radius of the sector is l and its central angle is $\frac{r}{l} \times 360^\circ$. So its area is

$$\frac{1}{360} \times \left(\frac{r}{l} \times 360\right) \times \pi l^2 = \pi r l$$

(Recall the computation of the area of a sector in Class 9).

Thus the area of the curved surface of a cone is half the product of the base circumference and the slant height.

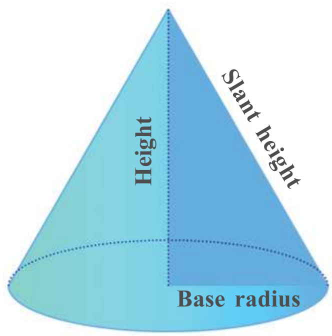
As in a pyramid, the height of a cone is the perpendicular distance from the apex to the base, and it is the distance between the apex and the centre of the base circle.



Again, as in the case of a square pyramid, the height is related to the slant height via a right triangle.

For example, in a cone of base radius 5 centimetres and height 10 centimetres the slant height is,

$\sqrt{5^2 + 10^2} = \sqrt{125} = 5\sqrt{5}$ centimetres



(1) What is the area of the curved surface of a cone of base radius 12 centimetres and slant height 25 centimetres?

(2) What is the surface area of a cone of base diameter 30 centimetres and height 40 centimetres?

(3) A conical fire work is of base diameter 10 centimetres and height 12 centimetres. 10000 such fireworks are to be wrapped in colour paper. The price of the colour paper is 2 rupees per square metre. What is the total cost?

(4) Prove that for a cone made by rolling up a semicircle, the area of the curved surface is twice the base area.

Volume of a cone

To find the volume of a cone, we can do an experiment similar to the one we did to find the volume of a square pyramid. Make a cone and a cylinder of the same base and height. Fill the cone with sand and transfer it to the cylinder. Here also, we can see that the volume of the cone is a third of the volume of the cylinder. Thus we have the following:

The volume of a cone is equal to a third of the product of the base area and height.

As in the case of square pyramids, draw a cylinder and a cone of same base and height in GeoGebra. Compare their volumes.

(A mathematical explanation of this also is given at the end of this lesson)

For example, the volume of a cone of base radius 4 centimetres and height 6 centimetres is

$$\frac{1}{3} \times \pi \times 4^2 \times 6 = 32\pi \text{ cubic centimetres.}$$

(1) The base radius and height of a cylindrical block of wood are 15 centimetres and 40 centimetres. What is the volume of the largest cone that can be carved out of this?

(2) The base radius and height of a solid metal cylinder are 12 centimetres and 20 centimetres. By melting it and recasting, how many cones of base radius 4 centimetres and height 5 centimetres can be made?

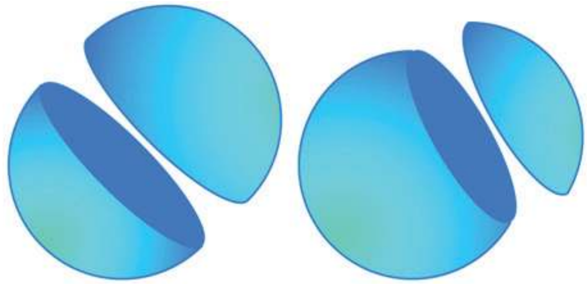
(3) A sector of central angle 216° is cut out from a circle of radius 25 centimetres and is rolled up into a cone. What are the base radius and height of the cone? What is its volume?

(4) The base radii of two cones are in the ratio 3 : 5 and their heights are in the ratio 2 : 3. What is the ratio of their volumes?

(5) Two cones have the same volume and their base radii are in the ratio 4 : 5. What is the ratio of their heights?

Sphere

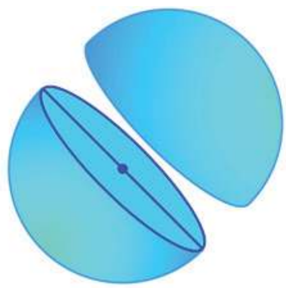
Round solids enter our lives in various ways - as the thrill of ball games and as the sweetness of laddus. Now let's look at the mathematics of such solids called *spheres*.



It we slice a cylinder or cone parallel to a base, we get a circle. In whatever way we slice a sphere, we get a circle.

The distance of any point on a circle from the centre is the same. A sphere also has a *centre*, from which the distance to any point on its surface is the same. This distance is called the *radius* of the sphere and double this is called the *diameter*.

If we slice a sphere into exact halves, we get a circle whose centre, radius and diameter are those of the sphere itself.



We cannot cut open a sphere and spread it flat, as we did with other solids. The fact is that we cannot make the surface of a sphere flat without some folding or stretching.

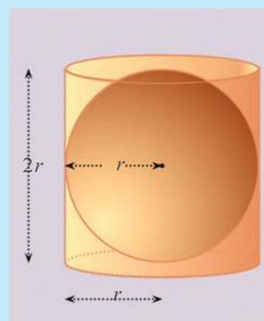
But we can prove that the surface area of a sphere of radius r is $4\pi r^2$ (An explanation is given at the end of the lesson).

The surface area of a sphere is equal to the square of its radius multiplied by 4π.

Also, we can prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ (An explanation of this also is given at the end of the lesson.)

Sphere and cylinder

Consider a cylinder which can cover a sphere precisely. Its base radius is the radius of the sphere and its height is double its radius.



So if we take the radius of the sphere as r, the base radius and height of the cylinder are r and $2r$.

So its surface area is

$$(2\pi r \times 2r) + (2 \times \pi r^2) = 6\pi r^2$$

The surface area of the sphere is $4\pi r^2$. Thus the ratio of these surface areas is 3 : 2.

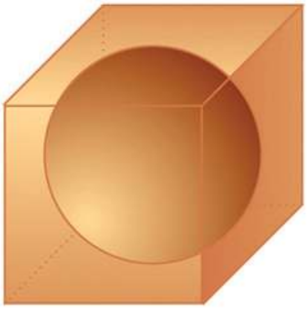
Again, the volume of the cylinder is

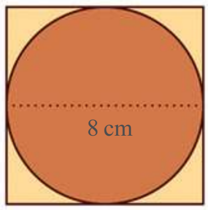
$$\pi r^2 \times 2r = 2\pi r^3$$

and the volume of the sphere is $\frac{4}{3}\pi r^3$, so that the ratio of the volumes is also 3 : 2

See this problem:

What is the surface area of the largest sphere that can be carved from a cube of edges 8 centimetres?



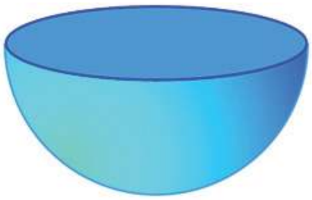


We can see from the picture that the diameter of the sphere is the length of an edge of the cube. So, the surface area of the sphere is

$$4\pi \times 4^2 = 64\pi \text{ square centimetres}$$

Another problem:

A solid sphere of radius 12 centimetres is cut into two equal halves. What is the surface area of each hemisphere?



The surface of the hemisphere consists of half the surface of the sphere and a circle.

Since the radius of the sphere is 12 centimetres, its area is,

$$4\pi \times 12^2 = 576\pi \text{ square centimetres}$$

Since the radius of the circle is 12 centimetres, its area is

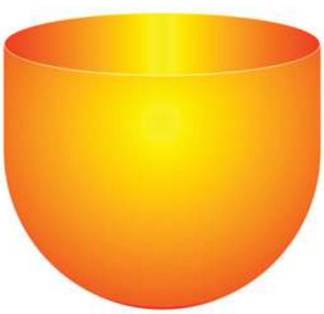
$$\pi \times 12^2 = 144\pi \text{ square centimetres}$$

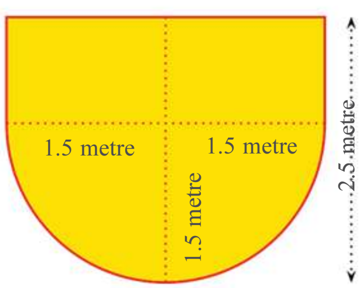
So the surface area of the hemisphere is,

$$\frac{1}{2} \times 576\pi + 144\pi = 432\pi \text{ square centimetres}$$

One more example:

A water tank is in the shape of a hemisphere attached to a cylinder. Its radius is 1.5 metres and the total height is 2.5 metres. How many litres of water can it hold?





The volume of the hemispherical part of the tank is,

$$\frac{2}{3}\pi \times (1.5)^3 = 2.25\pi \text{ cubic metres.}$$

8QL7PB

And the volume of the cylindrical part is

$$\pi \times (1.5)^2 (2.5 - 1.5) = 2.25\pi \text{ cubic metres.}$$

So, the total volume is

$$2.25\pi + 2.25\pi = 4.5\pi \approx 14.13 \text{ cubic metres.}$$

Since one cubic metre is 1000 litres, the tank can hold about 14130 litres of water.

(1) The surface area of a solid sphere is 120 square centimetres. If it is cut into two halves, what would be the surface area of each hemisphere?

(2) The volumes of two spheres are in the ratio 27 : 64. What is the ratio of their radii? And the ratio of their surface areas?

(3) The base radius and length of a metal cylinder are 4 centimetres and 10 centimetres. If it is melted and recast into spheres of radius 2 centimetres each, how many spheres can be made?

(4) A metal sphere of radius 12 centimetres is melted and recast into 27 small spheres of equal size. What is the radius of each small sphere?

(5) From a solid sphere of radius 10 centimetres, a cone of height 16 centimetres is carved out. What fraction of the volume of the sphere is the volume of the cone?

(6) The picture shows the dimensions of a petrol tank.

How many litres of petrol can it hold?

(7) A solid sphere is cut into two hemispheres. From one, a square pyramid and from the other a cone, each of maximum possible size are carved out. What is the ratio of their volumes?

8QV3QY

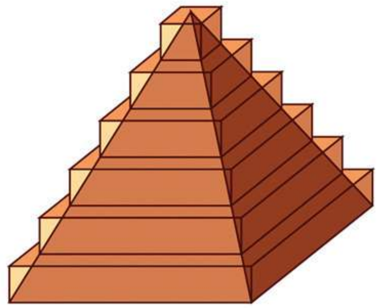
What is the speciality of the lateral faces of a square pyramid of maximum volume that can be cut out from a solid hemisphere?

Appendix

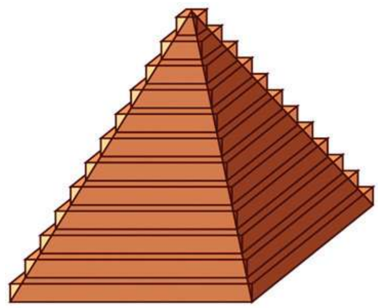
We have seen only the techniques of calculating volumes of pyramids and cones, and also the surface area and volume of a sphere. For those who may be interested in knowing how they are actually got, we give some explanations below.

Volume of a pyramid

We can think of a stack of square plates, of decreasing size as an approximation to a square pyramid.



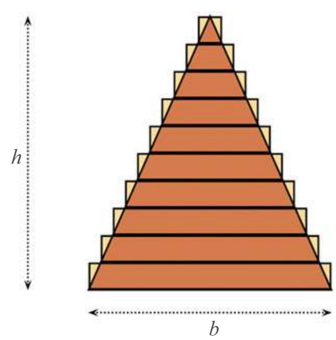
As we decrease the thickness of the plates and increase their number, we get better approximations.



And the sum of the volumes of these plates get nearer to the volume of the pyramid.

Suppose we use 10 plates, to start with. Each plate is a square prism of small height. Let's use plates of same height. So, if we take the height of the pyramid as h, each plate is of height $\frac{1}{10}h$. How do we compute the base of each plate?

If we imagine the pyramid and the stack of plates sliced vertically down from the vertex, we get a picture like this.



Starting from the top, we have isosceles triangles of increasing size. Their heights increase at the rate of $\frac{1}{10}h$ for each plate.

Since these triangles are all similar (why?) their bases also increase at the same rate. So, if we take the base edge of the bases of the pyramid to be b, the bases of the triangles starting from the top are $\frac{1}{10}b$, $\frac{2}{10}b$, ..., b.

So, the volumes of the plates are

$$\left(\frac{1}{10}b\right)^2 \times \frac{1}{10}h, \left(\frac{2}{10}b\right)^2 \times \frac{1}{10}h, \dots, b^2 \times \frac{1}{10}h$$

And their sum?

$$\frac{1}{10}b^2h\left(\frac{1}{10^2}+\frac{2^2}{10^2}+\dots+\frac{9^2}{10^2}+\frac{10^2}{10^2}\right) = \frac{1}{1000}b^2h\left(1^2+2^2+3^2+\dots+10^2\right)$$

We have seen how such sums can be computed in the section, **Sum of squares** of the lesson, **Arithmetic Sequences**.

$$1^2 + 2^2 + 3^2 + \dots + 10^2 = \frac{1}{6} \times 10 \times (10 + 1) \times (2 \times 10 + 1)$$

Thus the sum of the volumes

$$\frac{1}{1000}b^2h\times\frac{1}{6}\times10\times11\times21 = \frac{1}{6}b^2h\times\frac{10}{10}\times\frac{11}{10}\times\frac{21}{10} = \frac{1}{6}b^2h\times1.1\times2.1$$

Now imagine 100 such plates (we cannot draw it anyway.)

The thickness of a plate becomes $\frac{1}{100}h$ and the base edges would be $\frac{1}{100}b, \frac{2}{100}b, \frac{3}{100}b, \dots, b$. So the sum of the volumes would be

$$\frac{1}{100^3}b^2h\left(1^2+2^2+3^2+\dots+100^2\right) = \frac{1}{100^3}b^2h\times\frac{1}{6}\times100\times101\times201$$

$$= \frac{1}{6}b^2h\times\frac{100}{100}\times\frac{101}{100}\times\frac{210}{100}$$

$$= \frac{1}{6}b^2h\times1.01\times2.01$$

What if we increase the number of plates to 1000? Without going through detailed computations, we can see that the sum of volumes would be

$$\frac{1}{6}b^2h\times1.001\times2.001$$

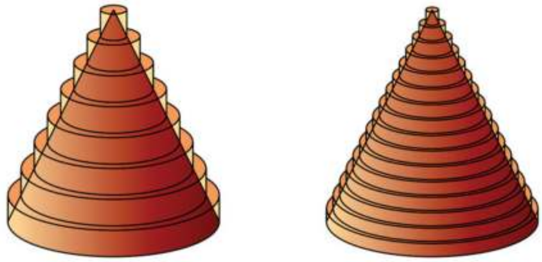
What is the number to which these sums get closer and closer to?

It is the volume of the pyramid; and it is

$$\frac{1}{6}b^2h\times1\times2 = \frac{1}{3}b^2h$$

Volume of a cone

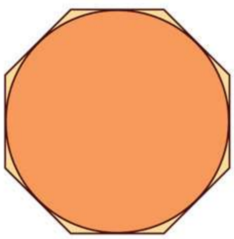
Just as we stacked square plates to approximate a pyramid, we can stack circular plates to approximate a cone.



And in much the same way, we can compute the volume of a cone also. (Try!)

Surface area of a sphere

First consider a circle through the middle of the sphere and a regular polygon with its sides touching it;

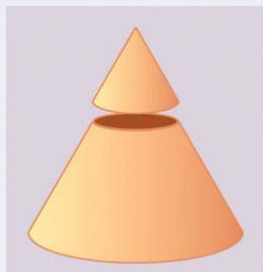


Now if this figure revolves, we get the sphere and a solid just covering it.

In the picture above, this solid can be split into two frustums and a cylinder.

Small and large

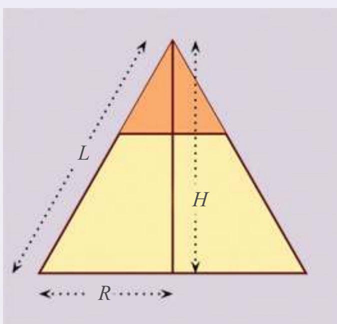
Cutting a cone parallel to the base, we get a small cone on top.

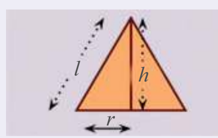


What is the relation between the dimensions of the small and large cones?

If we take the base radius, height and slant height of the large cone as R, H, L and those of the small cone as r, h, l, we get

$$\frac{r}{R} = \frac{h}{H} = \frac{l}{L}$$

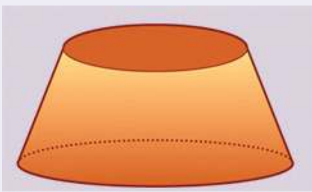




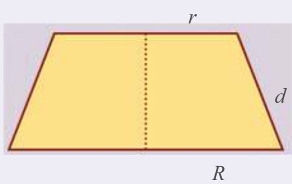
As we increase the number of sides of the polygon, the covering solid approximates the sphere better.

Frustum of a cone

If we cut off a small cone from the top of a cone, the remaining piece is called the frustum of a cone.



How do we find the area of the curved surface of a frustum in terms of its slant height and base radii?



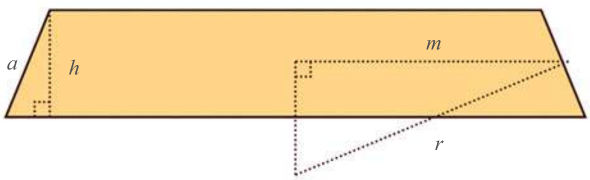
Taking the slant heights of the large and small cones as L and l, we get the d in the figure above as $d = L - l$. So the surface area of the frustum is,

$$\begin{aligned}\pi RL - \pi rl &= \pi(RL - rl)\\ &= \pi(R(l + d) - rl)\\ &= \pi(Rl + Rd - rl)\end{aligned}$$

Now, as seen earlier, we have $\frac{r}{R} = \frac{l}{L}$. So that $Rl = rL$. Using this, the area of the curved surface is,

$$\begin{aligned}\pi(rL + Rd - rl) &= \pi(r(L - l) + Rd)\\ &= \pi(rd + Rd)\\ &= \pi(r + R)d\end{aligned}$$

To compute the area of the curved surface of these frustums, let's consider one of these. Let's take its height as h and the radius of its middle circle as m. Let's also take the radius of the sphere as r and the length of a side of the covering polygon as a. We then have a figure like this.



The two right triangles in the figure are similar and so

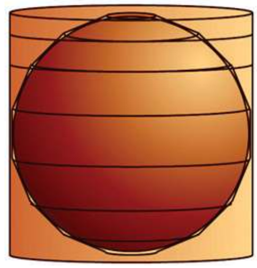
$$\frac{m}{r} = \frac{h}{a}$$

which can be written as

$$am = rh$$

The area of the curved surface of the frustum got by revolving this is $2\pi ma$, as shown in the side bar. **Frustum and cylinder**, on the last page. And this is equal to $2\pi rh$, by the above equation; that is, the area of the curved surface of a cylinder of base radius r and height h.

So what do we get? In the solid which approximates the sphere, the curved surface area of each frustum is equal to that of a cylinder of the same height with base radius equal to that of the sphere.



So the curved surface area of the whole approximating solid is equal to the sum of the curved surface areas of all these cylinders. And what do we get on putting together all these cylinders? A large cylinder, just covering the sphere.

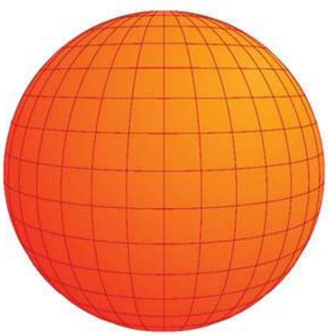
As we increase the number of sides of the polygon covering the circle, it becomes more circle-like; and the solid covering the sphere becomes more sphere-like. As seen just now, the curved surface area of any such solid is equal to the curved surface area of a cylinder just covering the sphere. So the surface area of the sphere is also equal to the area of the curved surface of this cylinder. Since the base radius of the cylinder is r and its height is $2r$, the area of its curved surface is

$$2\pi \times r \times 2r = 4\pi r^2$$

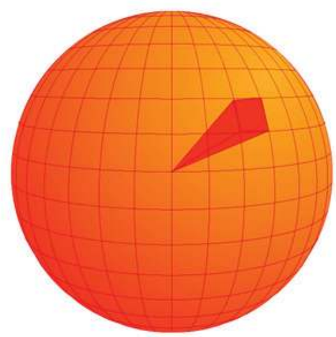
8R4YSL

Volume of a sphere

See these pictures:



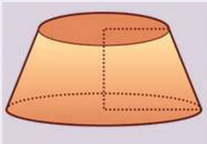
A sphere is divided into cells by horizontal and vertical circles. If we join the corners of such a cell to the centre of the sphere, we get a pyramid-like solid.



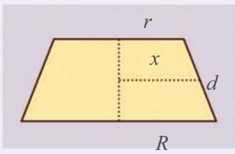
The sphere is made up of such solids joined together; and so the volume of the sphere is the sum of the volumes of these solids. Now if we change each cell into an actual square which touches the sphere, we get a solid which just covers the sphere; and the solid is made up of actual square pyramids. The heights of all these pyramids are equal to the radius of the sphere. If we take it as r and the base area of a pyramid as a, the volume of a pyramid is $\frac{1}{3}ar$.

Frustum and cylinder

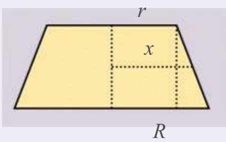
We have seen that the area of the curved surface of a frustum is $\pi(r+R)d$



Taking the radius of the circle round its middle as x, we get a figure like this:



Let's draw one more line:



From the two similar right triangles on the right,

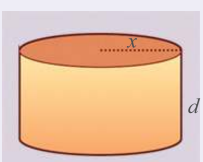
$$\frac{x-r}{R-r}=\frac{1}{2}$$

Simplifying, this gives

$$x=\frac{1}{2}(R+r)$$

So, the area of the curved surface of the frustum can be written $2\pi xd$.

But this is the area of the curved surface of a cylinder of base radius x and height d.



The volume of the solid covering the sphere is the sum of the volumes of all such pyramids. The bases of all such pyramids make up the surface of the solid, so that the sum of the base areas of the pyramids is the surface area of this solid. If it is taken as s, the volume of the solid would be $\frac{1}{3}sr$.

As we decrease the size of the cells and increase their number, the solid covering the sphere approximates the sphere better; and the surface area of the solid gets closer to the surface area of the sphere. Since the surface area of the sphere is $4\pi r^2$, the volume of the solid gets closer to the number

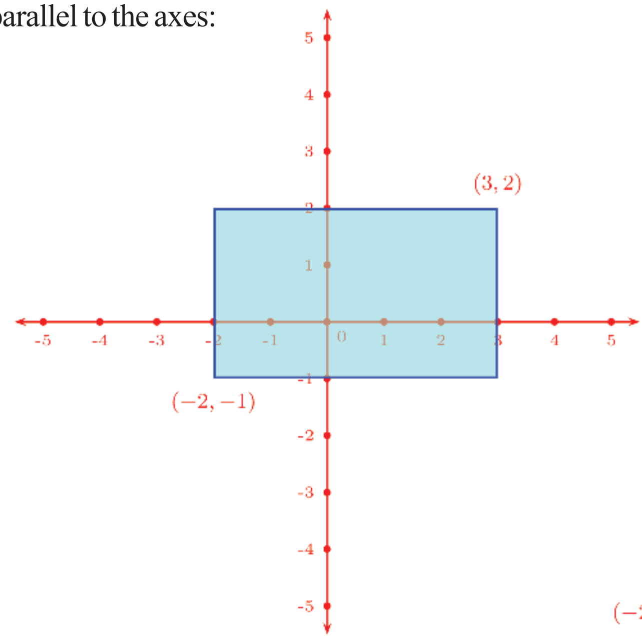
$$\frac{1}{3}\times 4\pi r^2\times r=\frac{4}{3}\pi r^3$$

And this is the volume of the sphere.

9 Geometry and Algebra

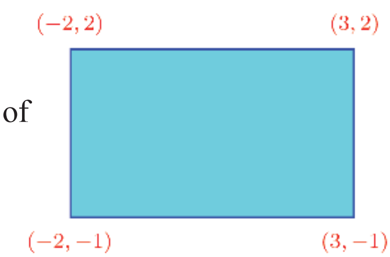
Triangles

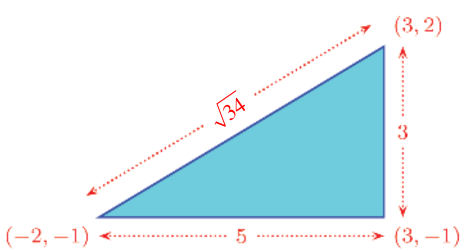
We have seen that if the line joining two points is not parallel to either axis, then we can draw a rectangle with these points as opposite vertices and sides parallel to the axes:



Mark a point A and type (x(A) + 3, y(A) + 2) in the Input Bar. What is the speciality of the coordinates of the new point got? Change the positions of A and check. What is the meaning of this command?

Moreover, we have seen how we can find the coordinates of the other two vertices without drawing the axes.





It was using such a rectangle that we computed the distance between two points like these, in terms of their coordinates. In fact, we didn't use the full rectangle, but only a right triangle forming half of it.

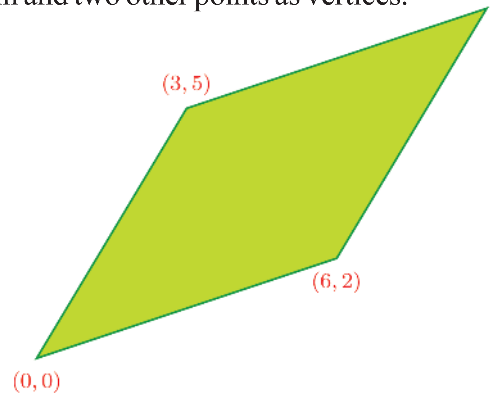
Algebra of geometry

We often write relations between numbers using algebra. Also we have seen how some of these relations between positive numbers can be described geometrically. If we represent points in a plane by pairs of numbers, we can write in algebra, the geometric relations between these points and also figures formed by them. An example of this is the fact that the fourth vertex of the parallelogram with vertices (0,0), (x_1, y_1), (x_2, y_2), is $(x_1 + x_2, y_1 + y_2)$.

Mark the point A(0,0) and two other points B, C. Type (x(B)+x(C), y(B)+y(C)) in the input bar to get the point D. Draw the quadrilateral ABCD. Isn't it a parallelogram? Why? Change the position of B and C and check.

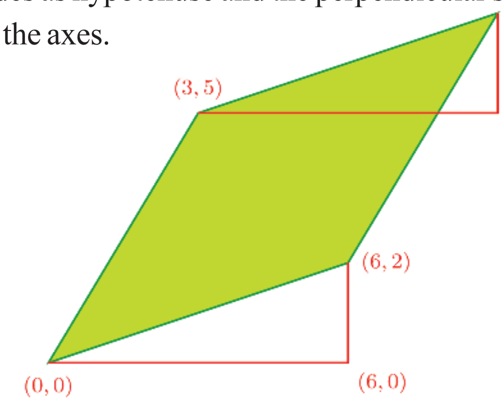
Draw triangle ABC and mark the coordinates of its vertices. Use the **Move** tool to shift the triangle to the right by 3. What happens to the coordinates of the vertices? Next shift the triangle up by 2. What happens to the coordinates now? What is the relation between the coordinates of the first and last triangles? Do this with a parallelogram instead of a triangle.

Computations using such right triangles are useful in many situations. For example, see this parallelogram with the origin and two other points as vertices.



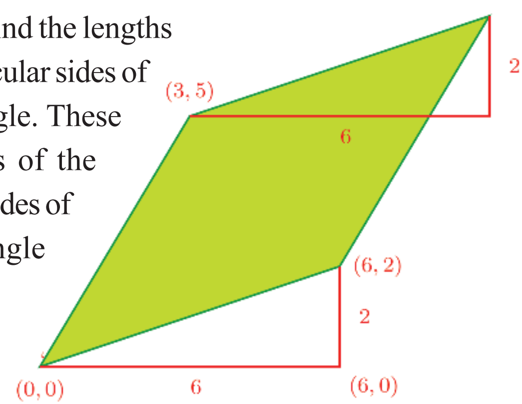
We have to find the fourth vertex.

For that, we draw right triangles with the top and bottom sides as hypotenuse and the perpendicular sides parallel to the axes.

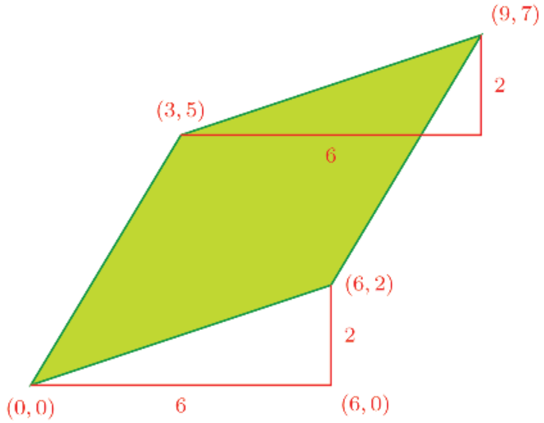


In these triangles, the hypotenuse and the angles at its ends are equal. (Why?) So, their perpendicular sides are also equal.

We can easily find the lengths of the perpendicular sides of the lower triangle. These are the lengths of the perpendicular sides of the upper triangle also.



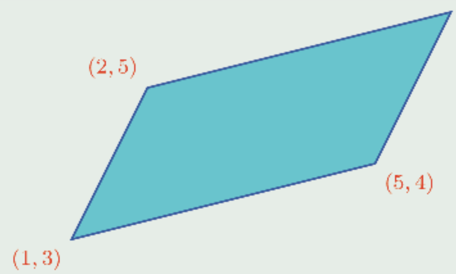
Now we can calculate the lower right corner of the upper triangle as (9, 5) and the top right corner as (9, 7). (How?)



To move a figure in geogebra we can use the **Translate by Vector** tool. Suppose we have drawn a parallelogram and want to shift it to 3 right and 2 up. For this first mark the points (0,0) and (3,2). Then choose the **Translate by Vector** tool and click on the parallelogram, the point (0,0) and point (3,2) in that order. What is the change in the coordinates of the parallelogram? Take other points instead of (3,2) and repeat.

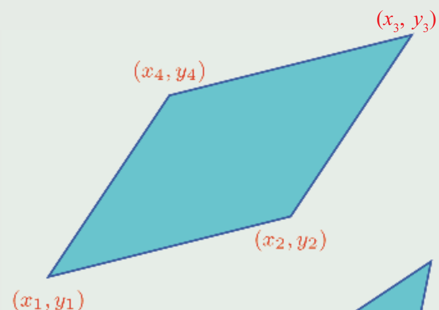
Mark three points A, B,C and get the fourth point D by typing (x(B)+x(C)−x(A), y(B)+y(C)−y(A)) in the input bar. Is ABCD a parallelogram? Why?

(1) What are the coordinates of the fourth vertex of the parallelogram shown on the right?

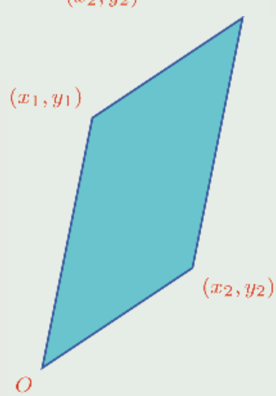


(2) The figure shows a parallelogram with the coordinates of its vertices:

Prove that $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$



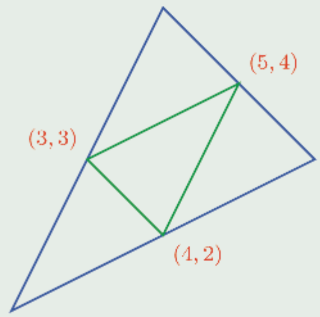
(3) A parallelogram is drawn with the lines joining (x_1, y_1) and (x_2, y_2) to the origin as adjacent sides. What are the coordinates of the fourth vertex?



(4) Prove that in any parallelogram, the sum of the squares of all sides is equal to the sum of the squares of the diagonals.

(5) In this picture, the mid points of the sides of the large triangle are joined to make a small triangle inside.

Calculate the coordinates of the vertices of the large triangle.



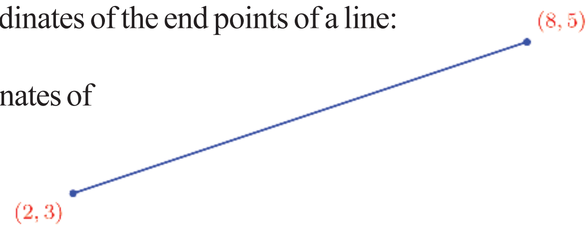
Midpoint

We have seen earlier that for a rectangle with sides parallel to the coordinate axes, if we know the coordinates of two opposite vertices, then we can find the coordinates of the other two (as in chapter **Coordinates**). Now we have seen that for any parallelogram, if we know the coordinates of three vertices, then we can find the coordinates of the fourth vertex.

Both these problems are about finding the coordinates of some points geometrically related to points with known coordinates. Let's look at another problem of this type.

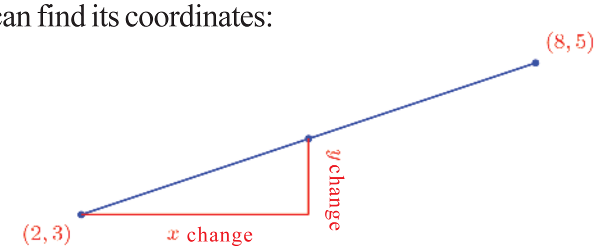
The figure shows the coordinates of the end points of a line:

We want to find the coordinates of the midpoint of this line.



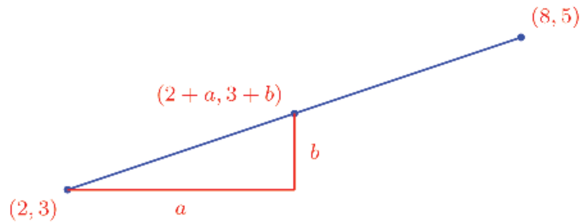
What we want is to find the point whose distance from (2, 3) is half the distance between the points (2,3) and (8, 5).

If we know the shifts in the x and y coordinates as we move from (2, 3) to this point, then we can find its coordinates:

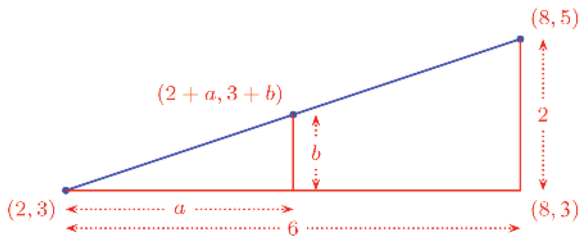


To get the midpoint of the line joining the points A and B, we type (A+B)/2 in the Input Bar

If we take these shifts as a and b, then we can write the coordinates of the midpoint as $(2 + a, 3 + b)$.



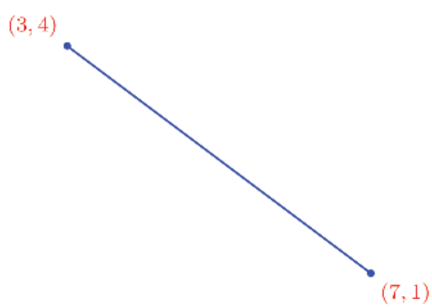
Let's mark also the shifts in coordinates as we move from (2, 3) to (8, 5):



In this figure, the large right triangle and the smaller one inside have the same angles (how?) So their sides are scaled by the same factor.

The hypotenuse of the smaller triangle is half that of the larger triangle; so the perpendicular sides are also halved. This means a and b are 3 and 1. Thus the midpoint is (5, 4)

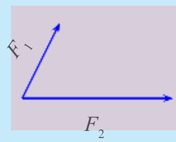
Now what if the line is like this?



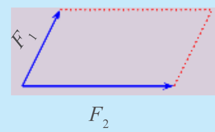
Force parallelogram

We can produce the same effect of two forces acting along different directions on a body, by a single force acting along a definite direction.

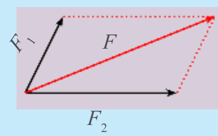
There is a method, recognized through experiments, to find this force and its direction. Draw two lines from a point with their lengths proportional to the forces (such as for example one centimetre for one Newton), along the directions of the forces:



Next draw a parallelogram with these as adjacent sides:

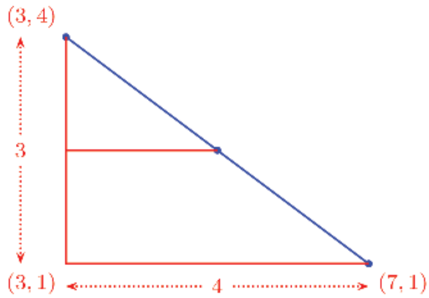


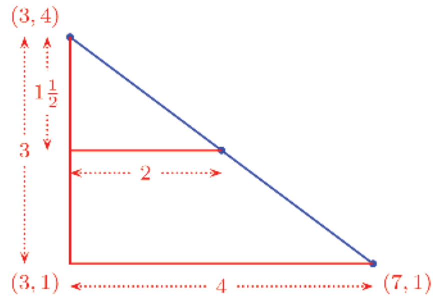
The single force to replace these two forces acts along the diagonal of this parallelogram; and its magnitude is the length of this diagonal, in the scale chosen.



This is known as the Parallelogram Law of Forces.

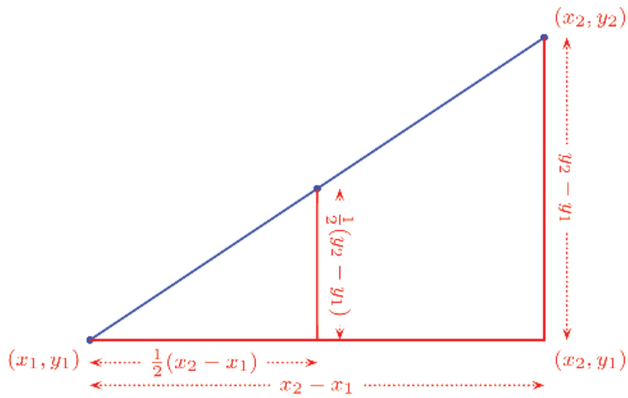
As we move from (3, 4) to (7, 1), the x-coordinate increases by 4 and the y coordinate decreases by 3. So, to reach the midpoint, the x coordinate must be increased by 2 and the y coordinate decreased by $1\frac{1}{2}$:





Thus the coordinates of the midpoint are $\left(5, 2\frac{1}{2}\right)$.

To find the midpoint without drawing any figure, let's take the coordinates of two points as (x_1, y_1) and (x_2, y_2), in general. Their relative positions can be of different types. First let's take them like this:



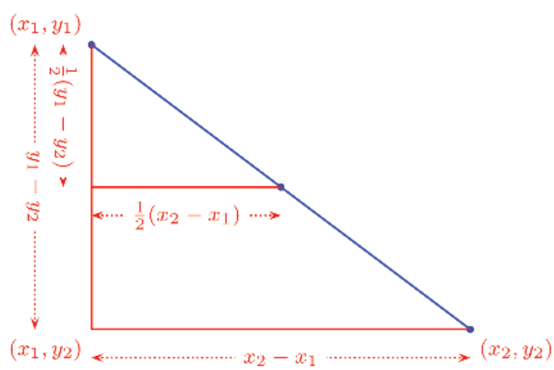
The x coordinate of the midpoint is

$$x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$$

and the y coordinate is

$$y_1 + \frac{1}{2}(y_2 - y_1) = \frac{1}{2}(y_1 + y_2)$$

What if the points are like this?



The x coordinate of the midpoint is

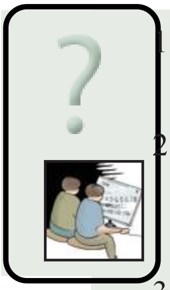
$$x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$$

and the y coordinate is

$$y_1 - \frac{1}{2}(y_1 - y_2) = \frac{1}{2}(y_1 + y_2)$$

Draw pictures for different relative positions of the points. What do you see?

The midpoint of the line **joining (x_1, y_1) and (x_2, y_2) is**
$$\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$$



1. A circle is drawn with the line joining (2, 3) and (6, 5) as diameter. What are the coordinates of the centre of the circle?

2. The coordinates of two opposite vertices of a parallelogram are (4,5) and (1, 3). What are the coordinates of the point of intersection of its diagonals?

3. The coordinates of the vertices of a quadrilateral, taken in order, are (2, 1), (5, 3), (8, 7), (4, 9).

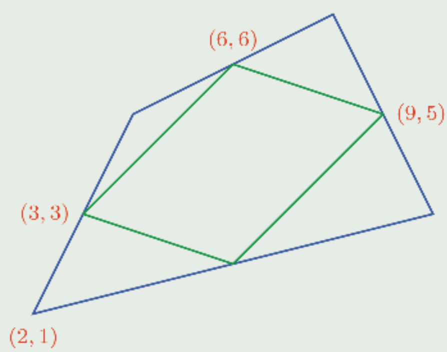
   (i) Find the coordinates of the midpoints of all four sides.

   (ii) Prove that the quadrilateral got by joining these midpoints is a parallelogram.

4. In the figure, the midpoints of the large quadrilateral are joined to form the smaller quadrilateral within:

   (i) Find the coordinates of the fourth vertex of the smaller quadrilateral.

   (ii) Find the coordinates of the other three vertices of the larger quadrilateral.

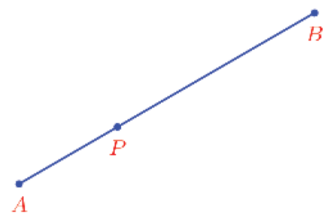


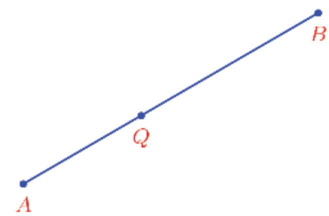
5. The coordinates of the vertices of a triangle are (3, 5), (9, 13), (10, 6). Prove that this triangle is isosceles. Calculate its area.

6. The centre of a circle is (1, 2) and a point on it is (3, 2). Find the coordinates of the other end of the diameter through this point.

Ratio

The midpoint of a line joining two points divides the line into two equal parts. Any other point on the line splits it into unequal parts. We can use ratios to specify such points. See these pictures:





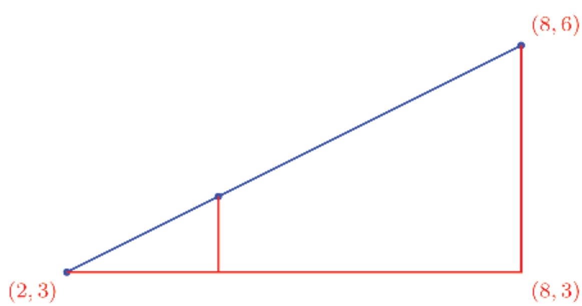
In the first picture, AP is $\frac{1}{3}$ of AB and the length of PB is the remaining $\frac{2}{3}$.

We say that the point P divides AB in the ratio 1 : 2.

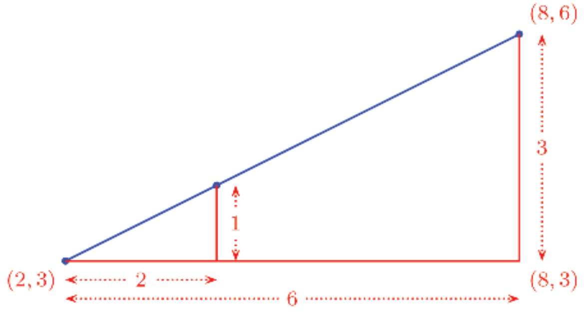
In the second picture, the point Q divides the line AB in the ratio 2 : 3. That is, AQ is $\frac{2}{5}$ of AB (and QB is the $\frac{3}{5}$ of AB).

To find the coordinates of the point which divides a line in a specified ratio, we can use the method we used to find the midpoint (with slight changes).

For example, let's find the coordinates of the point which divides the line joining (2, 3) and (8, 6) in the ratio 1 : 2. As in the case of the midpoint, we start by drawing right triangles:



In this picture, the hypotenuse of the smaller triangle is $\frac{1}{3}$ of the hypotenuse of the larger one. So, the perpendicular sides are also likewise. Hence we can mark their lengths like this:



Now we can calculate the coordinates of the required point as (4, 4).

What is our method of calculating these coordinates?

To reach (8, 6) from (2, 3), we must add 6 to the x coordinate and 3 to the y coordinate; so to reach $\frac{1}{3}$ of the distance between these points, we must add $\frac{1}{3}$ of these lengths.

Similarly, how do we calculate the coordinates of the point dividing the line joining the points (1,6) and (11, 2) in the ratio 3 : 5?

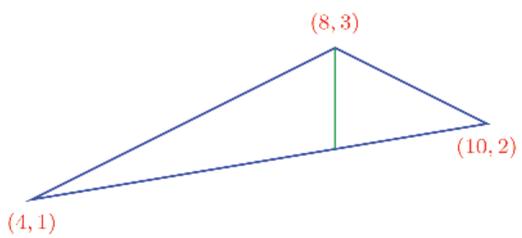
To reach (11, 2) from (1, 6), we must add 10 to the x coordinate and subtract 4 from the y coordinate. The point we need is $\frac{3}{8}$ of the length of the line away from (1, 6). Can't we calculate its coordinates without drawing a picture?

$$x \text{ coordinate} = 1 + 10 \times \frac{3}{8} = 4\frac{3}{4}$$

$$y \text{ coordinate} = 6 - 4 \times \frac{3}{8} = 4\frac{1}{2}$$

To get the point dividing the line joining A and B in the ratio 3 : 5, we type A+(3/8)(B-A) in the Input Bar

Now look at this picture:



The line inside the large triangle is the bisector of the top angle. We want to find the coordinates of the point where it meets the bottom side.

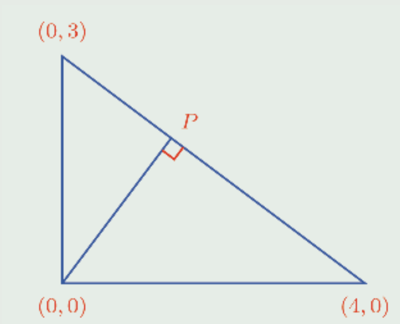
The bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle, isn't it? (The section **Triangle division** in the chapter **Area** of the Class 9 textbook.)

We can compute the lengths of the left and right sides of the triangle as $2\sqrt{5}$ and $\sqrt{5}$. So, what we need are the coordinates of the point which divides the line joining (4, 1) and (10, 2) in the ratio 2 : 1. We can compute this as before:

$$x \text{ coordinate } = 4 + (10 - 4) \times \frac{2}{3} = 8$$

$$y \text{ coordinate } = 1 + (2 - 1) \times \frac{2}{3} = 1\frac{2}{3}$$

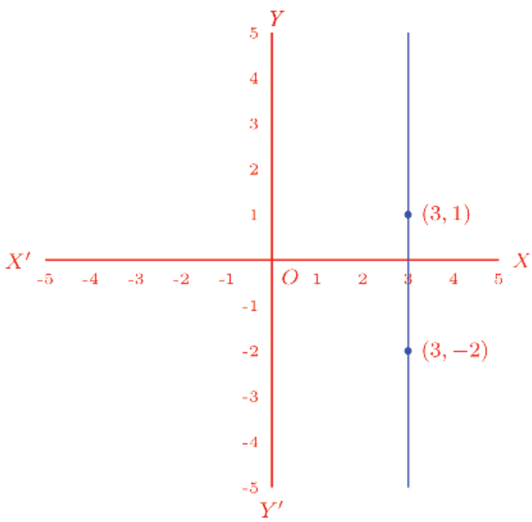
1. The coordinates of the points A and B are (3, 2) and (8, 7). Find the coordinates of
   (i) the point P on AB with $AP : PB = 2 : 3$
   (ii) the point Q on AB with $AQ : QB = 3 : 2$
2. Find the coordinates of the points which divide the line joining (1, 6) and (5, 2) into three equal parts.
3. The coordinates of the vertices of a triangle are (−1, 5), (3, 7), (1, 1). Find the coordinates of its centroid.
4. Calculate the coordinates of the point P in the picture:

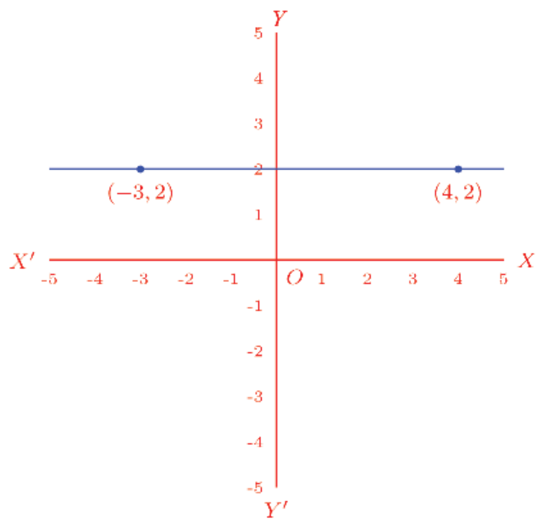


What are the coordinates of the centroid of the triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3)?

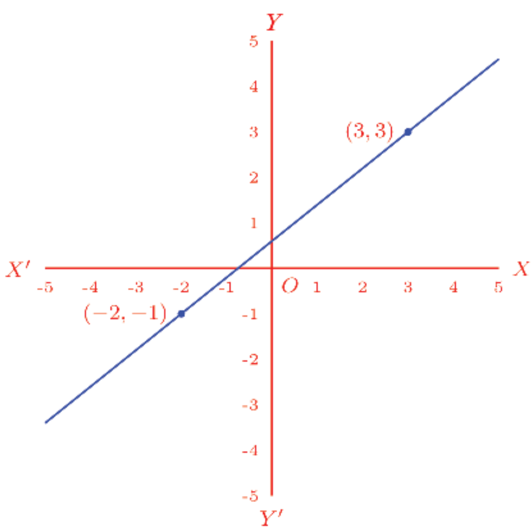
Straight line

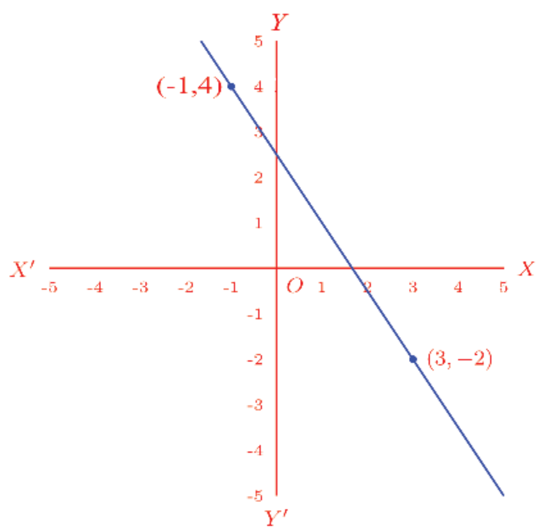
We can draw a straight line (and only one straight line) joining any two points. And we can extend it as much as we like to either side. If the x coordinates of the points are equal, then the line would be parallel to the y axis; and if the y coordinates are equal, it would be parallel to the x axis:



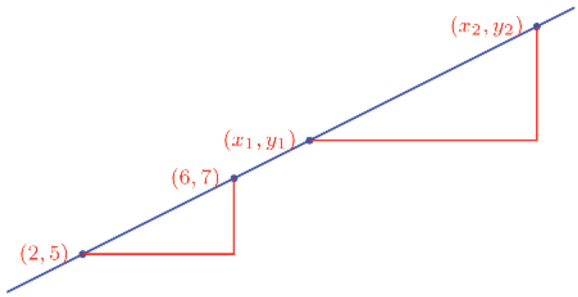


If both the x coordinates and the y coordinates are different, the line would be slanted, not parallel to either axis:

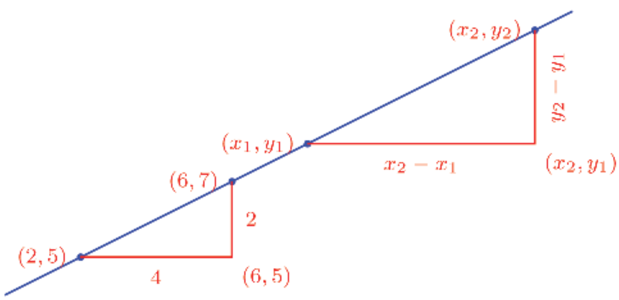




As we move along such a line, the x and y coordinates change at every point. There is a rule to this change. See this picture:



It shows the line joining the points (2, 5) and (6, 7) and two other points (x_1, y_1) and (x_2, y_2) on it; we have also drawn two right triangles with these two pieces of the line as hypotenuse and perpendicular sides parallel to the axes. We can mark the lengths of these perpendicular sides like this:



The perpendicular sides of the lower triangle are 4 and 2. That is, the vertical side is half the horizontal side. It must be so for the upper triangle also (why?) Thus

$$y_2 - y_1 = \frac{1}{2}(x_2 - x_1)$$

In this, (x_1, y_1) and (x_2, y_2) can be any two points on the line.

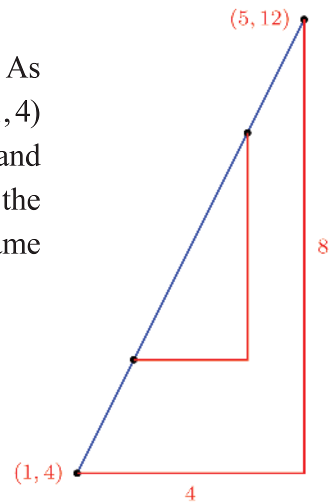
> For any two points on the line joining (2, 5) and (6, 7), the y difference is half the x difference

We can put this in a different way. As we move along this line from one point to another, the x and y coordinates change; and the rule for this change is this:

> In moving along the line joining (2, 5) and (6, 7), the change in y at every stage is half the change in x

What if we take some other pair of points, instead of (2, 5) and (6, 7)?

For example, let's take (1, 4) and (5, 12). As we move along the line joining them from (1, 4) to (5, 12), the x coordinate increases by 4 and the y coordinate by 8. In other words, the y change is twice the x change. And the same is true for any two positions on the line:

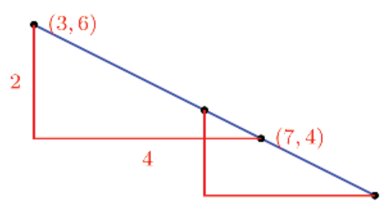


Thus

In moving along the line joining (1, 4) and (5, 12), the change in y at every stage is twice the change in x.

In both these lines, as x increases, so does y. It can be otherwise.

For example, if we take the points (3, 6) and (7, 4), as the x coordinate increases by 4, the y coordinate decreases by 2:



What do we see in general in all these?

In any line not parallel to either axis, the change in y coordinate is the product of the change in x coordinate with a fixed number

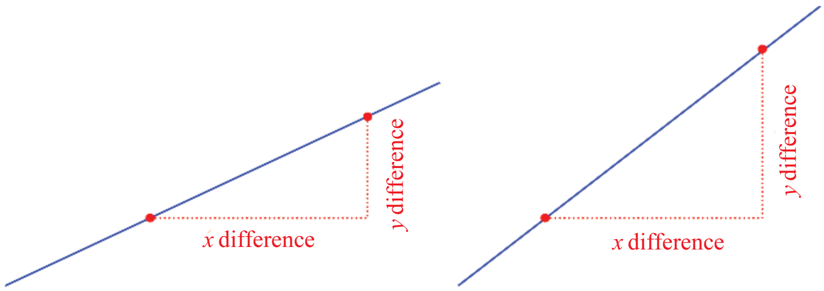
Do you remember the name for such a change?

In any line not parallel to either axis, the change in y is proportional to the change in x

Along a line parallel to the x axis, the y coordinate does not change at all; so the y difference of any two points on such a line is 0, and it is the x difference multiplied by 0. So, in this case also, the y difference is a fixed multiple of the x difference; but the x, y change is not proportional.

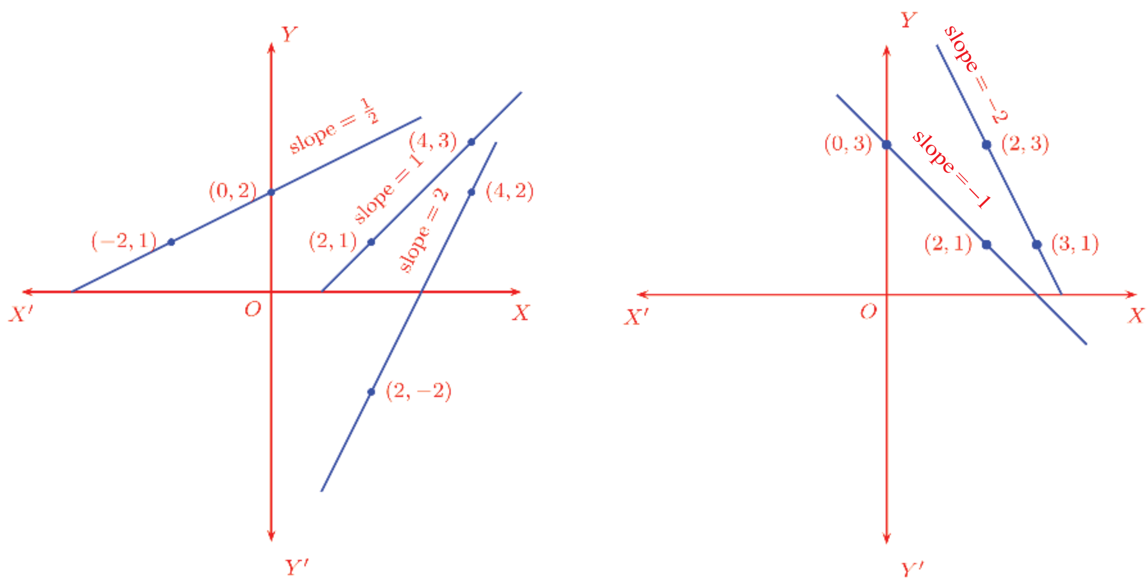
Make a slider named a and mark a point A. Type (x(A)+a,y(A)+2a) in the Input Bar. We get a new point. Set Animation On for the slider and see the path along which this point moves. We can set **Trace On** for the point to make it visible. Change the position of A and see what happens.

Geometrically, the x difference is the horizontal shift and the y difference is the vertical shift:



So, on dividing the y difference by the x difference, we get the rate of change of the vertical shift with respect to the horizontal shift.

In other words, the constant of proportionality of the change in coordinates of a line is a measure of the slant of the line. It is called the **slope** of the line:



We can use this idea to find other points on a line joining two specified points.

For example, let's look at the line joining (3, 5) and (6, 7). For these two points the x difference is 3 and the y difference is 2. So, anywhere on this line, as the x coordinate changes by 3, the y coordinate changes by 2. In other words, the y change is $\frac{2}{3}$ of the x change. Geometrically, we say that the slope of the line is $\frac{2}{3}$.

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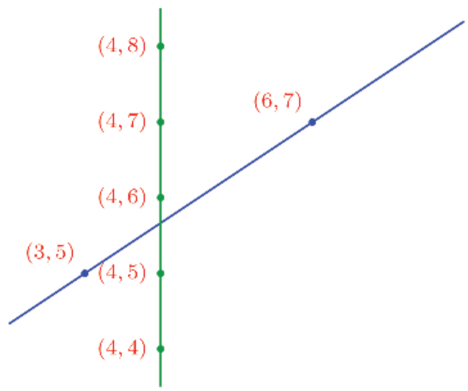
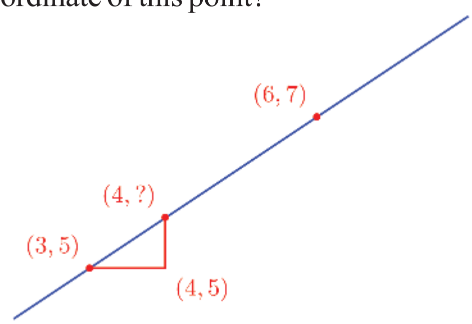
Now suppose we increase the x coordinate of the point (3, 5) by 1 and make it 4. Is there a point on this line with x coordinate 4? The line through (4, 5) parallel to the y axis cuts this line, doesn't it?



What is the y coordinate of this point?



Its x coordinate is 1 more than 3; so to get the y coordinate, we must add $\frac{2}{3}$ to 5. Thus $\left(4,\ 5\frac{2}{3}\right)$ is a point on this line. In the same way, we can find a point on this line with its x coordinate any number we choose.

For example, what is the point on this line with x coordinate 9?

9 is 6 added to 3; so to get the y coordinate, we must add $6 \times \frac{2}{3} = 4$ to 5. Thus (9, 9) is a point on this line.

To get the slope of a line in GeoGebra, choose **Slope** and click on the line

We have seen that the points (3, 5), (6, 7), (9, 9) are points on the same line. Is there any relation between the x coordinates 3, 6, 9 of these points? What about the y coordinates 5, 7, 9? Can you find some more points on this line with natural number coordinates?

Physics, algebra and geometry

Suppose a body moves such that the distance travelled is 10 metres in the first second, 15 metres in the next second, 20 metres in the second after and it goes on increasing like this. So, its speed also is increasing every second, as 10m/s during the first second, 15m/s during the next second, 20m/s during the second after that and so on.

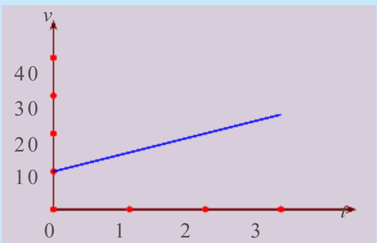
In other words, the speed increases by 5m/s every second. In the language of physics, the body has an acceleration of 5 metres per second per second (written 5metres/sec/sec or 5 m/s^2)

We can use the algebraic equation

$$v = 10 + 5t$$

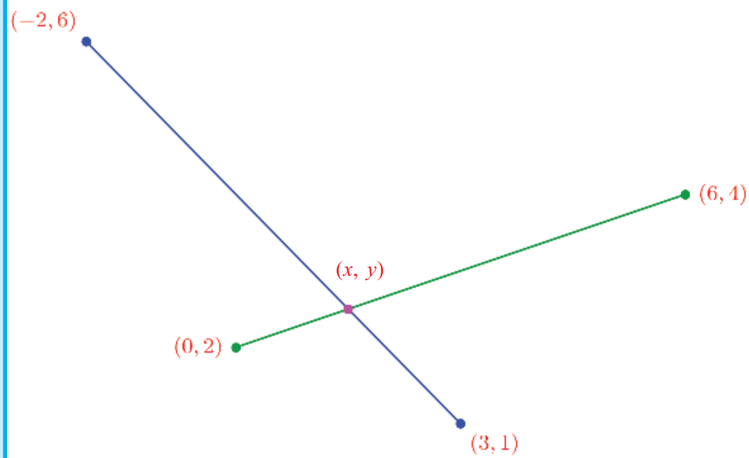
to calculate the speed v of this body at time t.

Now let's mark the various t and v along a pair of perpendicular lines and plot this relation between time and speed.



The slope of this line is 5. Here the slope is the rate at which v changes with respect to t; that is, the acceleration.

We can use the same idea to find the point of intersection of two lines. For example, let's take the line joining (0, 2) and (6, 4), and the line joining (3, 1) and (–2, 6). Let their point of intersection be (x, y):



Then the point (x, y) is on both the lines.

In the right leaning line, as the x coordinate changes from 0 to 6, the y coordinate changes from 2 to 4. That is, as the x coordinate increases by 6, the y coordinate increases by 2.

So, anywhere on this line, the y change is $\frac{1}{3}$ of the x change.

Since (x, y) is a point on this line, we get

$$y - 2 = \frac{1}{3}x$$

What about the left leaning line?

As the x coordinate increases by 5, the y coordinate decreases by 5. So, anywhere on this line, as the x coordinate increases by 1, the y coordinate decreases by 1. That is, the y change is the negative of the x change. Since (x, y) is a point on this line also, we get

$$y - 1 = 3 - x$$

Now if we rewrite the equations got from the two lines as

$$\begin{aligned} x - 3y &= -6 \\ x + y &= 4 \end{aligned}$$

Then as seen in the chapter **Pairs of Equations** of the Class 9 textbook, we can find the numbers x and y. Thus we get

$$x = 1\frac{1}{2} \qquad y = 2\frac{1}{2}$$

That is, the lines intersect at $\left(1\frac{1}{2},\ 2\frac{1}{2}\right)$

1. Prove that the points (1, 3), (2, 5) and (3, 7) are on the same line
2. Find the coordinates of two more points on the line joining (−1, 4) and (1, 2)
3. $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are arithmetic sequences. Prove that all the points with coordinates in the sequence $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ of number pairs, are on the same line
4. Prove that if the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are on a single line, then the points $(3x_1 + 2y_1, 3x_1 - 2y_1), (3x_2 + 2y_2, 3x_2 - 2y_2), (3x_3 + 2y_3, 3x_3 - 2y_3)$ are also on a single line. Would this be true if we take some other numbers instead of 3 and 2?

Figures and equations

For the coordinates of all points on the line joining the points (2, 4) and (5, 6), what is the relation between the x change and the y change?

For these two points, as the x coordinate increases by 3, the y coordinate increases by 2. That is, the y change is $\frac{2}{3}$ of the x change.

And this is true for all points on this line. So the relation between the changes in the x and y coordinates of any point (x, y) on this line and the point (2, 4) is this:

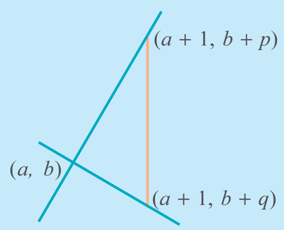
For the pair of points (1,3), (5,6) and for the pair of points (2,4), (6,7), the y difference divided by the x difference is $\frac{3}{4}$. Draw the line joining each pair of points in GeoGebra. Is there any relation between the lines?

Perpendicular slopes

We saw that the slopes of parallel lines are equal. What is the relation between the slopes of perpendicular lines?

Suppose that lines of slopes p and q are perpendicular to each other. Let's take their point of intersection as (a, b).

Then the point $(a + 1, b + p)$ is on the first line and the point $(a+1, b+q)$ is on the second line (why?)



Since the lines are perpendicular, the points (a, b), $(a + 1, b + p)$, $(a+1, b+q)$ are the vertices of a right angled triangle. The hypotenuse is the line joining the second and third points. So, the squares of the lengths of the perpendicular sides of this triangle are $p^2 + 1$ and $q^2 + 1$ and the length of the hypotenuse is $|p - q|$. This gives

$$(p^2 + 1) + (q^2 + 1) = (p - q)^2$$

Simplifying, this gives

$$2 = -2pq$$

which means

$$pq = -1$$

Thus, for lines perpendicular to each other, the slope of one is the negative of the reciprocal of the slope of the other.

$$y - 4 = \frac{2}{3}(x - 2)$$

We can rewrite this equation like this:

$$3(y - 4) = 2(x - 2)$$

We can simplify this further and write

$$2x - 3y + 8 = 0$$

What does this mean?

If we take any point on this line, its x and y coordinates satisfy this equation. That is, if the point with coordinates (p, q) is on this line, then $2p - 3q + 8 = 0$.

On the other hand, if we take a pair of numbers satisfying this equation, would the point with these numbers as coordinates be on this line?

For example, taking $x = 11$ and $y = 10$, we find

$$2x - 3y + 8 = 22 - 30 + 8 = 0$$

So, is the point (11, 10) on this line?

As seen in an earlier example, the line through (11, 4) parallel to the y axis intersects this line. The x coordinate of the point of intersection is 11; let's take the y coordinate as y.

We have noted that for any two points on this line, the y change is $\frac{2}{3}$ of the x change. Since (5,6) and (11, y) are points on the line, this gives

$$y - 6 = \frac{2}{3}(11 - 5) = 4$$

and this means $y = 10$. Thus (11, 10) is indeed a point on the line.

In the same way, we can see that if we take any pair x, y of numbers satisfying the equation $2x - 3y + 8 = 0$, the point (x, y) is on the line joining (2, 4) and (5, 6).

Typing 2x–3y+8=0 in the Input Bar of GeoGebra, we get the line with this as its equation. Make three sliders a, b, c and type ax+by+c=0 in the Input Bar. Change the values of a, b, c using the sliders and see the resulting changes in the line

Suppose that we have found a pair p and q of numbers such that $2p - 3q + 8 = 0$. If the point of intersection of the line joining (2, 4), (5, 6) and the line through $(p, 4)$ parallel to the y axis is taken as (p, y), then as in our example, we get

$$y - 4 = \frac{2}{3}(p - 2)$$

We can write this as

$$y = \frac{2}{3}(p - 2) + 4$$

Now from the equation $2p - 3q + 8 = 0$, we get

$$q = \frac{2}{3}(p - 2) + 4$$

Thus we get $y = q$ and this means the point (p, q) is on the line.

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What have we seen here?

> The set of coordinates of the points on the line joining the points (2, 4) and (5, 6), and the set of number pairs satisfying the equation $2x - 3y + 8 = 0$ are the same.

We shorten this as below:

The equation of the line joining the points (2, 4) and (5, 6) is $2x - 3y + 8 = 0$

Similarly, once we get the coordinates of two points on a line, we can write its equation.

For example, let's look at the line joining (0, 0) and (1, 1). The x difference and the y difference are equal for these points. It is so anywhere on this line. Thus for any point (x, y) on this line, the x and y differences from (0, 0) are equal. That is, $y = x$.

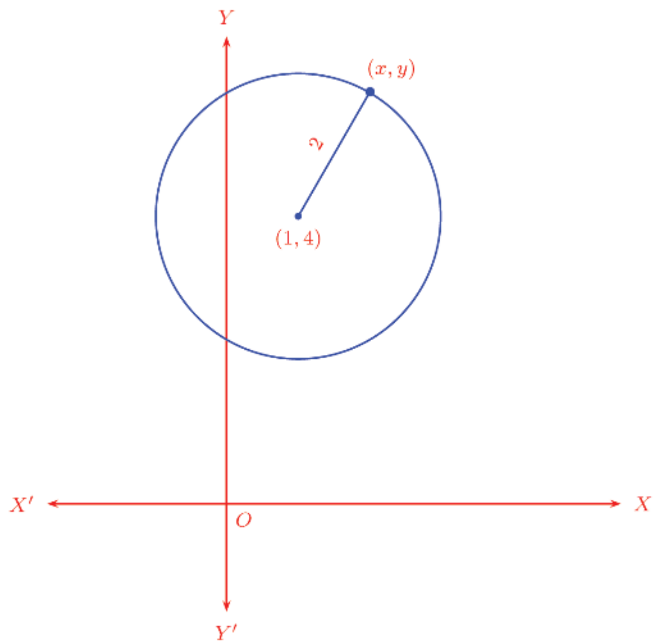
This is the equation of this line. (We can also write this as $x - y = 0$).

Draw the line joining (3, 5) and (6, 7). The equation of this line can be seen in the **Algebra View**.

We can draw circles in GeoGebra using the **Circle with Center and Radius** tool. Draw a circle with centre (1,4) and radius 2. We can see its equation in the **Algebra View**.

From the equation, we see that the x and y coordinates of any point on this line are equal.

We can write equations for not only straight lines but for other geometric figures also. For example, let's take the circle with centre at (1, 4) and radius 2. The distance between any point on the circle and the centre is 2:

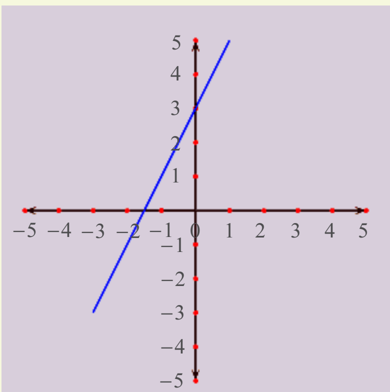


We have seen that the square of this distance is $(x-1)^2 + (y-4)^2$. Since this is the square of the radius of the circle, we get

$$(x-1)^2 + (y-4)^2 = 4$$

First degree plot

The plot of $y = 2x + 3$ done by a computer is shown below y:



Plot some more first degree polynomials. Do you get a straight line every time? To plot such a figure in GeoGebra, it is enough to give y= 2x+3 in the Input Bar.

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The x and y coordinates of any point on the circle satisfy this equation; on the other hand if we take any pair of numbers satisfying this equation, the point with those numbers as coordinates would be a point on this circle.

So, this is the equation of the circle. We can also expand (and then simplify) this and write it as

$$x^2 + y^2 - 2x - 8y + 13 = 0$$

What is the equation of the circle with centre at the origin and radius 1?

If we take the coordinates of a point on this circle as (x, y), the square of its distance from the origin is $x^2 + y^2$. Since this is equal to the square of the radius, we get

$$x^2 + y^2 = 1$$

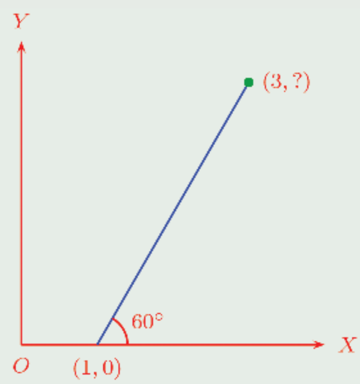
This is the equation of the circle.

(1) Find the equation of the line joining (1, 2) and (2, 4). For points on this line with consecutive natural numbers 3, 4, 5, ... as x coordinates, what is the sequence of y coordinates?

(2) Find the equation of the line joining (−1, 3) and (2, 5). Prove that if the point (x, y) is on this line, so is the point $(x + 3,\ y + 2)$.

(3) Prove that whatever number we take as x, the point $(x, 2x + 3)$ is a point on the line joining (−1, 1) and (2, 7).

(4) In the picture below, the x coordinate of a point on the slanted (blue) line is 3:



i) What is its y coordinate?

ii) What is the slope of the line?

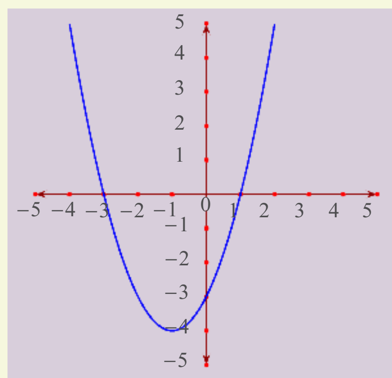
iii) Write the equation of the line.

Second degree plot

The plot below is that of the equation

$$y = x^2 + 2x - 3$$

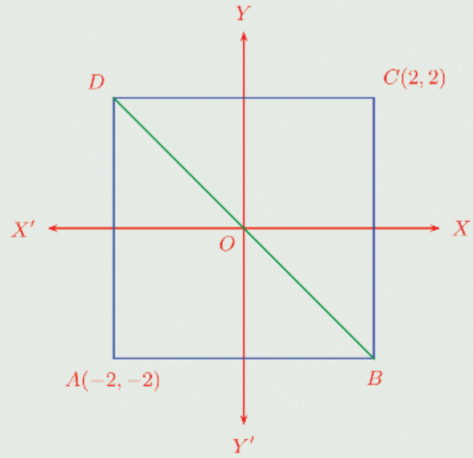
done using a computer:



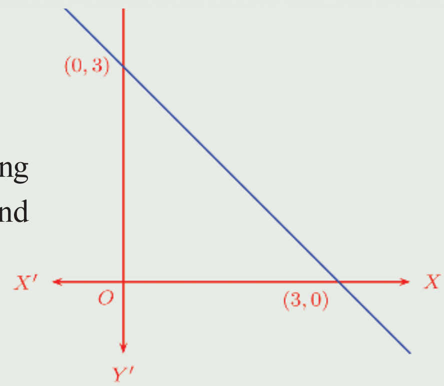
Plot a few more second degree polynomials like this.

To draw such a picture in GeoGebra type y=x^2+2x−3 in the Input Bar.

(5) In the picture here, $ABCD$ is a square: Prove that for any point on the diagonal BD, the sum of the x and y coordinates is zero.



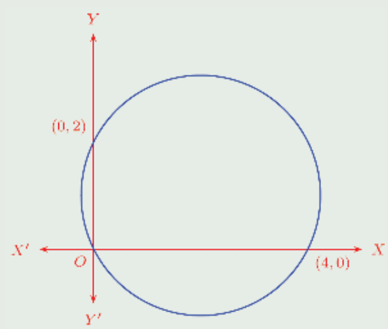
(6) Prove that for any point on the line intersecting the axes in the picture, the sum of the x and y coordinates is 3.



(7) Find the equation of the circle with centre at the orgin and radius 5. Write the coordinates of eight points on this circle.

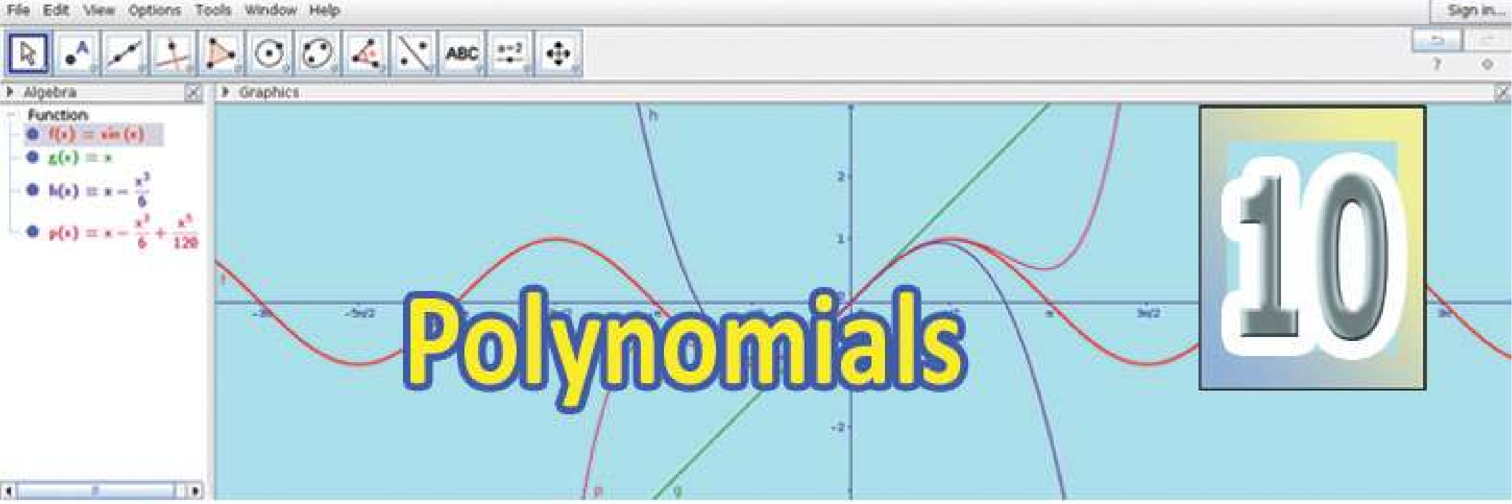
(8) Prove that if (x, y) be a point on the circle with the line joining $(0, 1)$ and $(2, 3)$ as diameter, then $x^2 + y^2 - 2x - 4y + 3 = 0$. Find the coordinates of the points where this circle cuts the y axis.

(9) What is the equation of the circle shown here?



If we type any equation connecting x and y in the Input Bar of GeoGebra, then we get the figure formed by the points with coordinates satisfying this equation. Try the following equations one by one:

- $2x$^2 + $2y$^2 = 4
- $2x$^2 + $3y$^2 = 4
- $2x$^2 − $3y$^2 = 4
- $2x$^2 + $3y$ = 4



Factors and solutions

We have seen in Class 8 that the difference of squares of two numbers is the product of the sum and difference of the numbers.

In the language of algebra,

$$x^2 - y^2 = (x + y)(x - y) \text{ for all numbers } x, y.$$

Let's take different numbers as y in this:

For any number x,

$$x^2 - 1 = (x + 1)(x - 1)$$

$$x^2 - 2 = \left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$$

$$x^2 - \frac{1}{4} = \left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$$

$x^2 - 1$, $x^2 - 2$, $x^2 - \frac{1}{4}$ are all second degree polynomials, while $x - 1$, $x + 1$, $x - \sqrt{2}$, $x + \sqrt{2}$, $x - \frac{1}{2}$, $x + \frac{1}{2}$ are all first degree polynomials.

Thus in all the equations above, a second degree polynomial is written as the product of two first degree polynomials.

When we write a number as the product of two numbers, those numbers multiplied are called factors. For example, since $12 = 2 \times 6$, we call 2 and 6, factors of 12. Similarly, since $x^2 - 1 = (x - 1)(x + 1)$, we call $x - 1$ and $x + 1$, *factors* of $x^2 - 1$.

Let's look at another example: we have seen in Class 8 how we can multiply the expressions $x + 2$ and $x + 3$ and write

$$(x + 2)\,(x + 3) = x^2 + 5x + 6$$

Writing this in reverse, we have

$$x^2 + 5x + 6 = (x + 2)\,(x + 3)$$

By what we have said just now, this means the first degree polynomials $x + 2$ and $x + 3$ are factors of the second degree polynomial $x^2 + 5x + 6$.

One more example: we know that

$$(x^2 + 1)\,(x + 2) = x^3 + 2x^2 + x + 2$$

Thus the second degree polynomial $x^2 + 1$ and the first degree polynomial $x + 2$ are factors of the third degree polynomial $x^3 + 2x^2 + x + 2$.

In general, we have the following:

If the polynomial $p(x)$ is the product of the polynomials $q(x)$ and $r(x)$, then we say that the polynomials $q(x)$ and $r(x)$ are factors of the polynomial $p(x)$

Now look at this product:

$$(x - 1)\,(x - 2) = x^2 - 2x - x + 2 = x^2 - 3x + 2$$

That is,

$$x^2 - 3x + 2 = (x - 1)\,(x - 2)$$

Thus we can *factorize* the polynomial $x^2 - 3x + 2$ into a product of first degree polynomials.

There is another thing to note here.

If we write $p\,(x) = x^2 - 3x + 2$, then what is $p(1)$?

We can compute it as

$$p\,(1) = 1 - 3 + 2 = 0$$

There's an easier way to do this. We have seen that

$$p\,(x) = (x - 1)\,(x - 2)$$

From this we have

$$p\,(1) = (1 - 1) \times (1 - 2) = 0 \times (-1) = 0$$

In the same way, can't we see that $p(2) = 0$ also?

Do we get $p(x) = 0$ for some other number x?

If $(x-1)(x-2) = 0$, then one of $x-1$ and $x-2$ must be zero, right?

Thus 1 and 2 are the two numbers we must take as x to get $p(x) = 0$. In other words, 1 and 2 are the solutions of the equation $p(x) = 0$ (that is, the equation $x^2 - 3x + 2 = 0$).

Let's look at another example: we have the product

$$(x-1)(x-2)(x-3) = (x^2 - 3x + 2)(x-3) = x^3 - 6x^2 + 11x - 6$$

Putting this in reverse, we say that $x-1$, $x-2$, $x-3$ are the first degree factors of the third degree polynomial $x^3 - 6x^2 + 11x - 6$. Here also if we write

$$p(x) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$$

we can see as in the first example that

$$p(1) = 0, p(2) = 0, p(3) = 0.$$

Thus, here also 1, 2, 3 are the solutions of the equation $p(x) = 0$ (that is, the equation $x^3 - 6x^2 + 11x - 6 = 0$).

We can use computers to do operations on not only numbers, but on algebraic expressions also. Programs which can do such operations are generally called Computer Algebra Systems (CAS). GeoGebra also has CAS capabilities.

What general principle do we get from these examples?

If the first degree polynomial $x - a$ is a factor of the polynomial $p(x)$, then $p(a) = 0$; that is, a is a solution of the equation $p(x) = 0$

We can put this in some more detail like this:

If the polynomial $p(x)$ can be split into first degree factors as

$$p(x) = (x - a_1)(x - a_2) \dots (x - a_n)$$

then the numbers $a_1, a_2, \dots, a_n$ are the solutions of the equation $p(x) = 0$

So, one method of solving a polynomial equation is to split the polynomial into a product of first degree factors.

We can use this to solve some second degree equations.

For example, look at this second degree equation:

$$x^2 - 5x + 6 = 0$$

We cannot write $x^2 - 5x + 6$ as a product of more than two first degree polynomials, can we? (The degree of the product of more than two first degree polynomials would be more than two, right?) So, we can try writing

$$x^2 - 5x + 6 = (x - a)(x - b)$$

Expanding the product, we get

$$x^2 - 5x + 6 = x^2 - (a + b)x + ab$$

The coefficients on both sides of the equation must be the same. For that, we must have

$$a + b = 5$$

$$ab = 6$$

That is, we must find two numbers with sum 5 and product 6.

A little thinking shows that 3 and 2 do the trick; thus we can take $a = 3$, $b = 2$ and write

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

From this, we can see that the solutions of the equation $x^2 - 5x + 6 = 0$ are 2 and 3.

Let's look at another equation:

$$x^2 + 2x - 15 = 0$$

As in the first problem, if we write

$$x^2 + 2x - 15 = (x - a)(x - b) = x^2 - (a + b)x + ab$$

then we get

$$a + b = -2$$

$$ab = -15$$

3 and 5 are factors of 15. Since the product is to be negative, we must have one of them negative. If we take -3 and 5, the sum won't be right; but 3 and -5 give the right sum. So taking $a = 3$ and $b = -5$ we can write

$$x^2 + 2x - 15 = (x - 3)(x + 5)$$

And we get the solutions of the equation $x^2 + 2x - 15 = 0$ as 3 and -5.

Write the second degree polynomials given below as the product of two first degree polynomials. Find also the solutions of the equation $p(x) = 0$ in each.

i)	$p(x) = x^2 - 7x + 12$	ii)	$p(x) = x^2 + 7x + 12$
iii)	$p(x) = x^2 - 8x + 12$	iv)	$p(x) = x^2 + 13x + 12$
v)	$p(x) = x^2 + 12x - 13$	vi)	$p(x) = x^2 - 12x - 13$

Factor theorems

We have seen that the first degree polynomial $x - 1$ is a factor of the polynomial $x^2 - 1$; also, if we take $x - 2$ instead of $x - 1$, then it is a factor of $x^2 - 4$ and if we take $x - \frac{1}{2}$, then it is a facor of $x^2 - \frac{1}{4}$. In general, $x - a$ is a factor of $x^2 - a^2$, whatever be the number a.

We can put this another way. In the second degree polynomial $p(x) = x^2$, if we take the number a as x, then $p(a) = a^2$, right? Then $p(x) - p(a) = x^2 - a^2$. Thus in the second degree polynomial $p(x) = x^2$, whatever number a we take as x, the first degree polynomial $x - a$ is a factor of the second degree polynomial $p(x) - p(a)$.

Would this be true if we take some other second degree polynomial, instead of x^2, as $p(x)$? For example, let's take

$$p(x) = 3x^2 + 2x - 1$$

Power difference

For any two numbers x and y, we can split $x^2 - y^2$ as a product:

$$x^2 - y^2 = (x - y)(x + y)$$

Similarly, we can write $x^3 - y^3$ like this:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

(Try it!). What about the difference of fourth powers? We can write

$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3)$$

Proceeding like this, for any natural number n, we can write $x^n - y^n$ as a multiple of $x - y$.

and a = 4. Then

$$p(a) = p(4) = (3 \times 4^2) + (2 \times 4) - 1$$

Let's compute $p(x) - p(a)$ without simplifying this:

$$p(x) - p(a) = (3x^2 + 2x - 1) - ((3 \times 4^2) + (2 \times 4) - 1)$$

How about combining the pairs x^2, 4^2 and x, 4 in this?

$$p(x) - p(a) = 3(x^2 - 4^2) + 2(x - 4)$$

We can write $x^2 - 4^2$ as $(x - 4)(x + 4)$. So, this equation becomes

$$p(x) - p(4) = 3(x - 4)(x + 4) + 2(x - 4)$$

In this equation, both terms in the sum contain $(x - 4)$.

Taking this as a common factor, we get

$$p(x) - p(4) = (x - 4)(3(x + 4) + 2)$$

This can be slightly simplified further to get

$$p(x) - p(4) = (x - 4)(3x + 14)$$

Thus we see that $x - 4$ is a factor of $p(x) - p(4)$

Let's look at one more example, taking $p(x) = 2x^2 + x - 4$ and $a = -2$. As in the first example, we first find

$$p(-2) = 2 \times (-2)^2 + (-2) - 4$$

Without simplifying this completely, we write

$$p(-2) = (2 \times 4) - 2 - 4$$

and subtract it from $p(x)$ to get

$$p(x) - p(-2) = (2x^2 + x - 4) - ((2 \times 4) - 2 - 4)$$

Pairing as in the first example, we get

$$p(x) - p(-2) = 2(x^2 - 4) + (x + 2)$$

The expression $x^2 - 4$ in this can be written $(x + 2)(x - 2)$. This makes the equation

$$p(x) - p(-2) = 2(x + 2)(x - 2) + (x + 2)$$

Taking $x + 2$ as a common factor in the sum on the right side, this becomes

$$p(x) - p(-2) = (x + 2)(2(x - 2) + 1) = (x + 2)(2x - 3)$$

Thus, $x + 2 = x - (-2)$ is a factor of $p(x) - p(-2)$

First degree factors

For any two numbers x and y, we have

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Taking different numbers as y in this, we get various results on polynomials such as

$x - 1$ is a factor of $x^3 - 1$

$x - 2$ is a factor of $x^3 - 8$

$x - \frac{1}{2}$ is a factor of $x^3 - \frac{1}{8}$

$x + 3$ is a factor of $x^3 + 27$

In other words, whatever be the number a, the polynomial $x - a$ is a factor of $x^3 - a^3$.

We can see that this is true for powers greater than 3 also, from the box, **Power difference**. In general

> Whatever be the number a, the polynomial $x - a$ is a factor of the polynomial $x^n - a^n$.

Using this we can prove a general theorem, as in the case of second degree polynomials:

> For any polynomial $p(x)$ and any number a, the polynomial $x - a$ is a factor of the polynomial $p(x) - p(a)$.

From this, we also get the theorem below:

> For any polynomial $p(x)$ and any number a, if $p(a) = 0$ then the polynomial $x - a$ is a factor of the polynomial $p(x)$.

In the same way we can see that for any second degree polynomial $p(x)$ and any number a, the polynomial $x - a$ is a factor of the polynomial $p(x) - p(a)$.

For those who are not convinced yet, we write this procedure completely in algebra. Let's take

$$p(x) = lx^2 + mx + n$$

and let a be any number. Then

$$\begin{aligned} p(x) - p(a) &= (lx^2 + mx + n) - (la^2 + ma + n) \\ &= l(x^2 - a^2) + m(x - a) \\ &= l(x - a)(x + a) + m(x - a) \\ &= (x - a)(l(x + a) + m) \\ &= (x - a)(lx + (la + m)) \end{aligned}$$

which shows that $x - a$ is a factor of $p(x) - p(a)$. Thus we have the general result:

> For any second degree polynomial $p(x)$ and for any number a, the polynomial $x - a$ is a factor of the polynomial $p(x) - p(a)$.

Suppose we take $p(x) = x^2 - 5x + 6$ and $a = 3$. Then $p(3) = 9 - 15 + 6 = 0$, so that $p(x) - p(3) = p(x)$. So, according to the general result above, $x - 3$ is a factor of $p(x)$ itself. This gives the general result below:

For any second degree polynomial $p(x)$ and any number a, $x - a$ is a factor of $p(x) - p(a)$. If $p(a)$ is zero we get $p(x) - p(a) = p(x)$. So $x - a$ is a factor of $p(x)$ itself.

In general,

For any second degree polynomial $p(x)$ and for any number a, if $p(a) = 0$, then the first degree polynomial $x - a$ is a factor of the polynomial $p(x)$

For example, if we take $p(x) = 3x^2 - 5x - 2$ and $a = 2$, then

$$p(a) = p(2) = 12 - 10 - 2 = 0$$

so that $x - 2$ is a factor of $3x^2 - 5x - 2$.

Polynomial division

Results on multiplication of numbers can also be stated as results of division. For example $2 \times 5 = 10$ can also be stated as $10 \div 2 = 5$ or $10 \div 5 = 2$. We can also write this in the form of fractions as $\frac{10}{2} = 5$ or $\frac{10}{5} = 2$.

Similary, the multiplication of polynomials

$$(x - 1)(x + 1) = x^2 - 1$$

can also be written in the form of fractions as

$$\frac{x^2 - 1}{x - 1} = x + 1$$

or

$$\frac{x^2 - 1}{x + 1} = x - 1$$

But we must be careful about one thing here. The equation $x^2 - 1 = (x - 1)(x + 1)$ is true, whatever number we take as x. But in the equation

$$\frac{x^2 - 1}{x - 1} = x + 1$$

we cannot take $x = 1$ (why?)

1) In each pair of polynomials given below, find the number to be subtracted from the first to get a polynomial for which the second is a factor. Find also the second factor of the polynomial got on subtracting the number.

   (i) $x^2 - 3x + 5, x - 4$

   (ii) $x^2 - 3x + 5, x + 4$

   (iii) $x^2 + 5x - 7, x - 1$

   (iv) $x^2 - 4x - 3, x - 1$

2) In the polynomial $x^2 + kx + 6$, what number must be taken as k to get a polynomial for which $x - 1$ is a factor? Find also the other factor of that polynomial.

3) In the polynomial $kx^2 + 2x - 5$, what number must be taken as k to get a polynomial for which $x - 1$ is a factor?

Quotient and remainder

We have seen that if we take a polynomial $p(x)$ and a number a, then $x - a$ is a factor of $p(x) - p(a)$. (The box, First degree factors).

So, the polynomial $p(x) - p(a)$ can be written as the product of $x - a$ and a polynomial $q(x)$:

$$p(x) - p(a) = (x - a)\ q(x)$$

We can make a slight change and write this as

$$p(x) = (x - a)\ q(x) + p(a)$$

This means that for any polynomial $p(x)$ and any number a, we can write $p(x)$ as a sum of a product of $x - a$ by a polynomial and a number.

This is somewhat like writing

$$18 = (7 \times 2)\ + 4$$

in the quotient-remainder form. So, in the equation $p(x) = (x - a)\ q(x) + p(a)$ also, $q(x)$ is called the quotient on dividing $p(x)$ by $x - a$ and $p(a)$ is called the remainder.

Solutions and factors

We have seen in the first section, how we can use factorization to solve certain second degree equations. On the other hand, we can use solutions of a second degree equation for factorization also. For example look at the polynomial

$$p(x) = x^2 - 30x + 221$$

To factorize this as in the first section, we must find two numbers with sum 30 and product 221.

There's another way. If $x - a$ is a factor of $p(x)$, then $p(a) = 0$, right?

That is, we must have $a^2 - 30a + 221 = 0$. In other words, a must be a solution of the equation

$$x^2 - 30x + 221 = 0$$

We know how to solve it.

$$x = \frac{30 \pm \sqrt{30^2 - 884}}{2} = \frac{1}{2}\ (30 \pm 4) = 17 \text{ or } 13$$

This means $p(17) = 0$ and $p(13) = 0$

From this, it follows that $x - 17$ and $x - 13$ are factors of $p(x)$.

Thus,

$$x^2 - 30x + 221 = (x - 17)(x - 13)$$

Let's look at another example. We want to factorize $x^2 - 2x - 1$ into first degree polynomials. For this we need only find the solutions of the equation $x^2 - 2x - 1 = 0$. These we can find as

$$x = \frac{2 \pm \sqrt{4+4}}{2} = \frac{1}{2}(2 \pm 2\sqrt{2}) = 1 \pm \sqrt{2}$$

Thus the solutions of the equation $x^2 - 2x - 1 = 0$ are $1 + \sqrt{2}$ and $1 - \sqrt{2}$. So, by the general result we have stated, $x - (1 + \sqrt{2})$ and $x - (1 - \sqrt{2})$ are factors of $x^2 - 2x - 1$. By actual multiplication, we can see that

$$x^2 - 2x - 1 = (x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$$

One more example: how do we write $2x^2 - 7x + 6$ as the product of two first degree polynomials?

Algebra in GeoGebra

With GeoGebra, not only can we draw geometrical shapes, but do some algebra also. (GeoGebra is a combination of Geometry and Algebra.)

To do algebra, open the CAS view in GeoGebra (View→CAS). As an example, if we type

(x–a)*(x^2+a*x+a^2), then we get $-a^3 + x^3$ as the answer. If we type solve (x^2–x–1=0), then we get

$$\left\{ x = \frac{-\sqrt{5}+1}{2},\ x = \frac{\sqrt{5}+1}{2} \right\}$$

Writing $p(x) = 2x^2 - 7x + 6$, we must solve the equation $p(x) = 0$ to find its first degree factors. The solutions of $2x^2 - 7x + 6 = 0$ are

$$x = \frac{7 \pm \sqrt{49-48}}{4} = \frac{1}{4}(7 \pm 1) = 2 \text{ or } \frac{3}{2}$$

So $x - 2$ and $x - \frac{3}{2}$ are factors of $2x^2 - 7x + 6$

Do we get $2x^2 - 7x + 6$ on multiplying these two?

$$(x - 2)\left(x - \frac{3}{2}\right) = x^2 - \frac{7}{2}x + 3$$

What do we do to change $x^2 - \frac{7}{2}x + 3$ to $2x^2 - 7x + 6$?

Thus

$$2x^2 - 7x + 6 = 2\left(x^2 - \frac{7}{2}x + 3\right) = 2(x - 2)\left(x - \frac{3}{2}\right)$$

If we want, we can also write it in the prettier form

$$2x^2 - 7x + 6 = (x - 2)(2x - 3)$$

Not all second degree polynomials can be factorized like this. For example, look at the polynomial $x^2 + 1$. The equation $x^2 + 1 = 0$ has no solutions (why?) Nor does the polynomial $x^2 + 1$ have any first degree factors.

1) Write the second degree polynomials given below as the product of two first degree polynomials:

   (i) $x^2 - 20x + 91$

   (ii) $x^2 - 20x + 51$

   (iii) $x^2 + 5x - 84$

   (iv) $4x^2 - 16x + 15$

   (v) $x^2 - x - 1$

2) Prove that none of the polynomials below can be factored into a product of first degree polynomials:

   (i) $x^2 + x + 1$

   (ii) $x^2 - x + 1$

   (iii) $x^2 + 2x + 2$

   (iv) $x^2 + 4x + 5$

3) In the polynomial $p(x) = x^2 + 4x + k$, upto what number can we take as k, so that $p(x)$ can be factorized as a product of two first degree polynomials?

11 Statistics

Not a correct average

The monthly income of 10 households in a neighbourhood are these:

16500	21700	18600	21050	19500
17000	21000	18000	22000	17500

What is the mean monthly income?

Adding all these and dividing by 10, we get the mean monthly income as 19285 rupees.

Now, if instead of taking all these incomes separately, we had only the mean, then also we can make some conclusions about the general economic status of the households:

- The monthly income of all these households are around 19285 rupees.
- None of the households has a monthly income very much greater or very much less than 19285 rupees.
- The number of households with monthly income greater than 19285 rupees is more or less equal to the number of households with monthly income less than 19285 rupees.

Now suppose someone with a monthly income of 175000 rupees comes to live in the neighbourhood. What is the mean monthly income of the 11 households?

$$\frac{(19285 \times 10) + 175000}{11} \approx 33441 \text{ rupees.}$$

Without giving all these details, if this mean only is given, wouldn't we make the wrong conclusion that all these households have a monthly income around 30000 rupees? This is almost one and a half times the monthly income of ten of these households.

The purpose of calculating the mean is to reduce a whole collection of numbers to a single number, which gives a general understanding of a situation. But numbers in the collection which are very much less or very much more than others (though few) affect the mean a lot.

In our example, it was a single number very much larger than the first ten which changed the mean so much. Can you think of other instances like this where very small or very large numbers influence the mean to give a wrong impression?

Another average

Let's see how we can compute another average which gives a better overall indication of the monthly income of the 11 households. If we write all the incomes in increasing order and take the middle number, 5 of the households would have income less than this and 5 of them would have more.

Let's write the numbers in order:

16500, 17000, 17500, 18000, 18600, 19500, 21000,
21050, 21700, 22000, 175000

The middle number is 19500. It is called the *median* of these numbers. That is, the median monthly income of the 11 households is 19500 rupees. We can put it like this: of all the 11 households, 5 have monthly income less than 19500 rupees and 5 have more than 19500 rupees. That is, the number of households with income less than the median and the number of households with income more than the median are equal.

What if we take only the first 10 households? If we write incomes in the increasing order, there would be two numbers, 18600 and 19500 at the middle, instead of just one number.

Here also, we must choose the median such that the number of items below it and above it are equal. Any number between 18600 and 19500 would do for this. Usually half their sum is taken as the median.

That is, the median monthly income of the first 10 households is

$\frac{1}{2}$ (18600 + 19500) = 19050 rupees

The median income 19050 rupees, like the mean income 19285 rupees gives a reasonable estimate of the economic status of the first ten households (and there is no great difference between the mean and the median either).

What is important here is that the high income of the eleventh household does not change the median much. Also if we say that the median income of some households is 19050 rupees and that the monthly income of one of these is 21000 rupees, we can conclude that this household is better off than more than half the households considered.

?

(1) The distance covered by an athlete in long jump practice are

6.10, 6.20, 6.18, 6.20, 6.25, 6.21, 6.15, 6.10

in metres. Find the mean and median. Why is it that there is not much difference between these?

(2) The table below gives the rainfall during one week of September 2015 in various districts of Kerala.

District	Rainfall (mm)
Kasaragod	66.7
Kannur	56.9
Kozhikode	33.5
Wayanad	20.5
Malappuram	13.5
Palakkad	56.9
Thrissur	53.4
Ernakulam	70.6
Kottayam	50.3
Idukki	30.5
Pathanamthitta	56.4
Alapuzha	45.5
Kollam	56.3
Thiruvananthapuram	89.0

Calculate the mean and median rainfall in Kerala during this week. Why is the mean less than median ?

(3) Prove that for a set of numbers in arithmetic sequence, the mean and median are equal.

Frequency and median

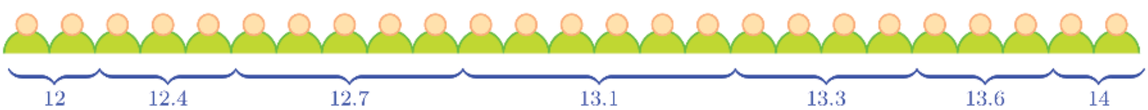
The amount of haemoglobin in blood is usually given as grams per decilitre (that is, 100 millilitres). The table below shows 25 children sorted according to haemoglobin levels, after a blood test.

Haemoglobin (g/dl)	Number of children
12.0	2
12.4	3
12.7	5
13.1	6
13.3	4
13.6	3
14.0	2

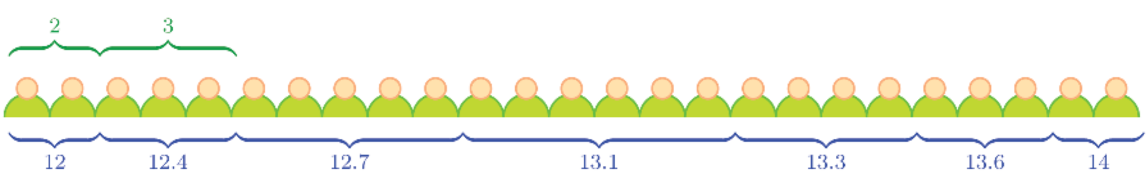
From this, we can compute the mean haemoglobin level. How do we find the median?

The median is that which comes in the middle; that is 12 of the children should have haemoglobin level less than the median level and 12 more than the median level.

To find it, we need only make the kids stand in a line, in the order of haemoglobin level and take the level of the thirteenth kid. Imagine the kids standing like this;

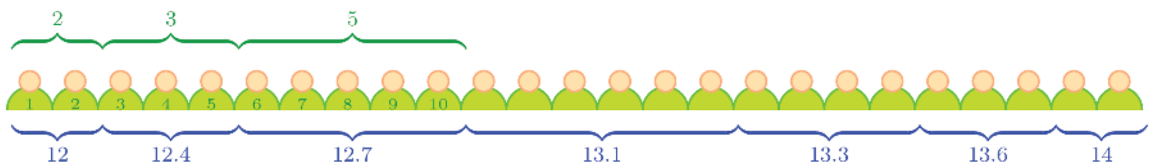


The first two have haemogloblin 12, the next 3 have 12.4 and thus the line grows.



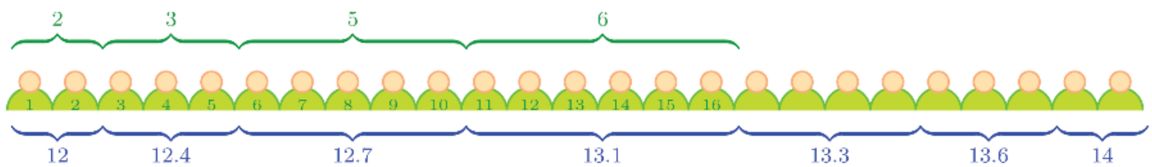
We want the haemoglobin level of the 13^{th} kid. By adding the numbers in the table one by one, we can find his position in the haemoglobin sequence. Taking the $2 + 3 = 5$ kids of the first two groups, the level rises to 12.4. That is the 5^{th} kid has level 12.4.

Adding the 5 kids in the next group, we have 5 + 5 = 10 kids and the level reaches 12.7.

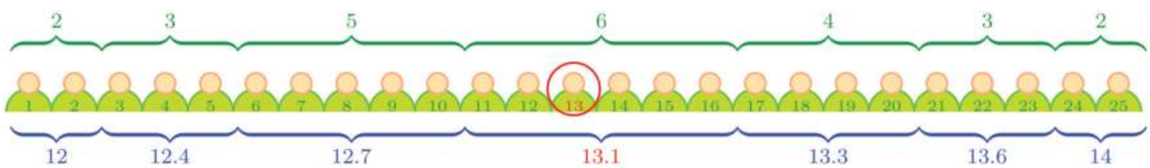


Thus the 10th kid is at level 12.7.

Adding the 6 kids in the next group, we have 10 + 6 = 16 kids.



We need the level of the 13th kid. The level of all the kids from the 11th to the 16th in the line is 13.1. So the 13th kid also has this level, and this is the median level.



We can do this using a table instead of a picture:

Haemoglobin (g/dL)	Number of children
up to 12.0	2
up to 12.4	5
up to 12.7	10
up to 13.1	16
up to 13.3	20
up to 13.6	23
up to 14.0	25

From the table, we can see the haemoglobin levels of the kids, from the 11th to the 16th, is 13.1. Since the middle one of 25, that is the 13th, is in this set, the median level can be found as 13.1.

(1) 35 households in a neighbourhood are sorted according to their monthly income in the table below.

Monthly income (Rs)	Number of households
4000	3
5000	7
6000	8
7000	5
8000	5
9000	4
10000	3

Calculate the median income.

(2) The table below shows the workers in a factory sorted according to their daily wages:

Daily wages (Rs)	Number of workers
400	2
500	4
600	5
700	7
800	5
900	4
1000	3

Calculate the median daily wage.

(3) The table below gives the number of babies born in a hospital during a week, sorted according to their birth weight.

Weight (kg)	Number of babies
2.500	4
2.600	6
2.750	8
2.800	10
3.000	12
3.150	10
3.250	8
3.300	7
3.500	5

Calculate the median birth-weight.

Classes and median

The table below shows the workers of a factory sorted according to their wages:

Daily Wages (Rs)	Number of Workers
400-500	6
500-600	7
600-700	10
700-800	9
800-900	5
900-1000	4
Total	**41**

How do we compute the median daily wage in this factory?

What we want to calculate is the daily wage of the worker in the middle, when the workers are arranged in order from the one earning the least to the one earning the most. Here there are 41 workers in all; so the one in the middle is the 21^{st} in this order.

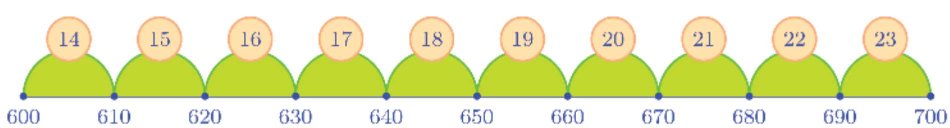
The table splits the wages into different classes. Let's first find the class to which the 21st person belongs. As in our previous problems, let's calculate the total when each class is joined to all classes before it:

Daily Wages (Rs)	Number of Workers
Below 500	6
Below 600	13
Below 700	23
Below 800	32
Below 900	37
Below 1000	41

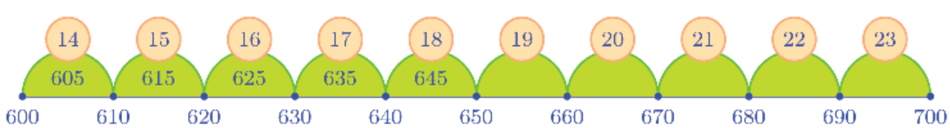
We see from the table that when we take together all those earning less than 600 rupees a day, then we reach upto the 13th person; and when we take together all those earning less than 700 rupees a day, then we reach upto the 23rd person. The 21st person we seek is between them. Thus we find that his daily wage is between 600 and 700 rupees.

What do we do to make this more exact? We only know that the 10 workers from the 14th to the 23rd earn between 600 and 700 rupees; we don't know what the actual earning of each is.

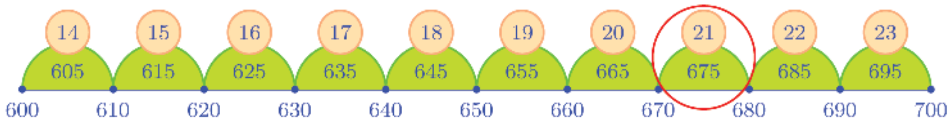
So, we need to make some assumptions. (We made some assumptions when we calculated the mean from a table involving classes and frequencies, remember?) We divide the 100 rupees from 600 to 700 into 10 equal parts and assume that each of these subdivisions contains exactly one person:



We further suppose that the daily wage of each of these workers is the midvalue of the class to which he belongs:



Under these assumptions, we can compute the daily wage of the 21st worker:



We can calculate this without drawing pictures. How did we compute the median wage, on the basis of our assumptions regarding the arrangement of workers?

- The daily wage of the 14^{th} worker is 605rupees
- The daily wage of each one thereafter, till the 23^{rd} increases by 10 rupees
- To reach the 21^{st} worker we want, from the 14^{th}, we must take 7 more workers

Now it is an arithmetic problem, isn't it?

What number do we get if we start from 605 and add 10 repeatedly 7 times?

We can calculate it as

$$605 + (7 \times 10) = 675$$

Haven't we seen many such problems in the chapter, **Arithmetic Sequences**?

Let's do one such problem, without drawing pictures. The table below shows the employees in an office, sorted according to their age:

Age	Number of Workers
25 – 30	4
30 – 35	7
35 – 40	8
40 – 45	10
45 – 50	9
50 – 55	8
Total	**46**

We want to compute the median age. Here the total number is 46, which is an even number. So we arrange the employees from the youngest to the oldest and take half the sum of the ages of the 23^{rd} and the 24^{th} persons as the median age. First let's write the cumulated frequencies:

Age	Number of Workers
Below 30	4
Below 35	11
Below 40	19
Below 45	29
Below 50	38
Below 55	46

According to this, the ages of the 10 persons from the 20th to the 29th positions in the order of ages, are between 40 and 45. The persons in the 23rd and 24th position whom we want are in this group.

As in the previous problem, we divide the 5 years between 40 and 45 into 10 equal parts, and assume that each such subdivision contains one person whose age is the midvalue of his class.

Then each subdivision is $\frac{5}{10} = \frac{1}{2}$ year. The age of the 20th person is the midvalue of 40 and $40\frac{1}{2}$, which is $40\frac{1}{4}$. Our assumption means the age of each succeeding person increases by $\frac{1}{2}$ year (till the 29th). So the age of the 23rd person is

$$40\frac{1}{4} + \left(3 \times \frac{1}{2}\right) = 40\frac{1}{4} + 1\frac{1}{2} = 41\frac{3}{4}$$

and the age of the 24th person is

$$41\frac{3}{4} + \frac{1}{2} = 42\frac{1}{4}$$

Now to get the median age, we take half the sum of these two:

$$\frac{1}{2}\left(41\frac{3}{4} + 42\frac{1}{4}\right) = \frac{1}{2} \times 84 = 42$$

So the median age is 42. In this example, the persons in the 23rd and 24th positions are in the same class 40 – 45. In contexts where this is not so, we compute the median in a slightly different manner. See this problem.

The table below shows children of a class sorted according to their marks in an exam:

Marks	Number of Children
0 – 10	4
10 – 20	7
20 – 30	9
30 – 40	12
40 – 50	8
Total	**40**

We want to compute the median mark.

First let's write the cumulated frequencies:

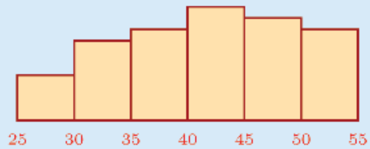
Marks	Number of Children
10	4
20	11
30	20
40	32
50	40

Here, if we arrange the children from the one with the least mark to the one with the greatest, then the number of children with marks below 30 is 20, which is exactly half the total number of 40 children.

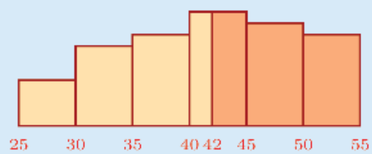
In this case, we take 30 itself as the median mark. The justification for this is that half the total number of children have marks below 30 and half of them have marks 30 or more.

Median area

Do you remember drawing histograms of frequency tables? The histogram of the age problem is like this:



In this, the vertical line through the median splits the picture into two parts:



It is not difficult to see that the areas of the two parts are equal (try it!)

Does the median have this property in all such computations?

(1) The table shows some households sorted according to their usage of electricity:

Electricity usage (units)	Number of households
80 – 90	3
90 – 100	6
100 – 110	7
110 – 120	10
120 – 130	9
130 – 140	4

Calculate the median usage of electricity

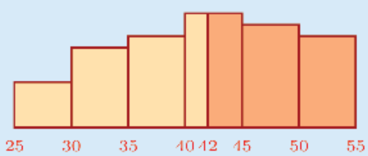
(2) The table below shows the children in a class sorted according to their marks in the math exam:

Marks	No. of Children
0 – 10	4
10 – 20	8
20 – 30	10
30 – 40	9
40 – 50	5

Calculate the median mark of the class.

Median probability

We have noted that the perpendicular through the median divides the histogram into two parts of equal area:



So, if we mark a point on this picture, the probability of it falling on either part is the same (that is the probability is $\frac{1}{2}$).

This means that in the office, given in the problem, if we select someone without any special consideration, the probability of his age being less than 42 or more than 42 is the same.

(3) The table below gives the details of the income tax paid by the employees in an office in a year:

Income tax (Rs)	Number of employees
1000 – 2000	8
2000 – 3000	10
3000 – 4000	15
4000 – 5000	20
5000 – 6000	22
6000 – 7000	8
7000 – 8000	6
8000 – 9000	3

Calculate the median income tax paid.

Let's know about cyber safety

There is absolutely no need to mention the advantages of Internet and Social Networking sites. We have embraced their potential for communication, entertainment and information seeking.

But over the period, it is seen that a lot of teenagers are being harassed and fall prey to the abuse of Social Media. You can easily prevent yourself from being a victim, if you take a few precautionary measures while being online.

How Social Networking sites can be dangerous

- Sharing and posting too much of personal information such as phone number, address, location, photos, etc., can be misused.
- Trusting strangers believing their profile to be true can be dangerous, as they may not be the same as stated.
- Snapshots of chats, photos, videos, etc., are saved and will be used for blackmailing and threatening.
- Being cyber bullied by posting negative, derogatory comments, posts, photos, etc. to tarnish one's image.
- Lots of predators and adult criminals are lurking online to trap children.

Tips for safe Social Networking

- Always keep your personal information strictly personal.
- Customize your privacy settings so that others can see only the basic information.
- Just know about and manage your friends. Don't trust all the online friends.
- Let your friends know that you are uncomfortable if they post something inappropriate about you.
- Do not publish any information that reveals your identity.
- Always use strong passwords. Don't share them with others.
- Never share your pictures, photographs, email accounts, etc., with anyone.
- Keep your personal messages strictly personal. Once posted they are published for ever.
- If ever threatened or bullied seek the help of parents/teachers.

Helpline Phone Numbers

Crime Stopper: 1090

Cyber Cell (Tvm): 9497975998

Control Room: 100

Child Helpline: 1098 / 1517